Quantum Error Correction
&
Holography

Workshop on Dynamics and Scrambling of Quantum Information

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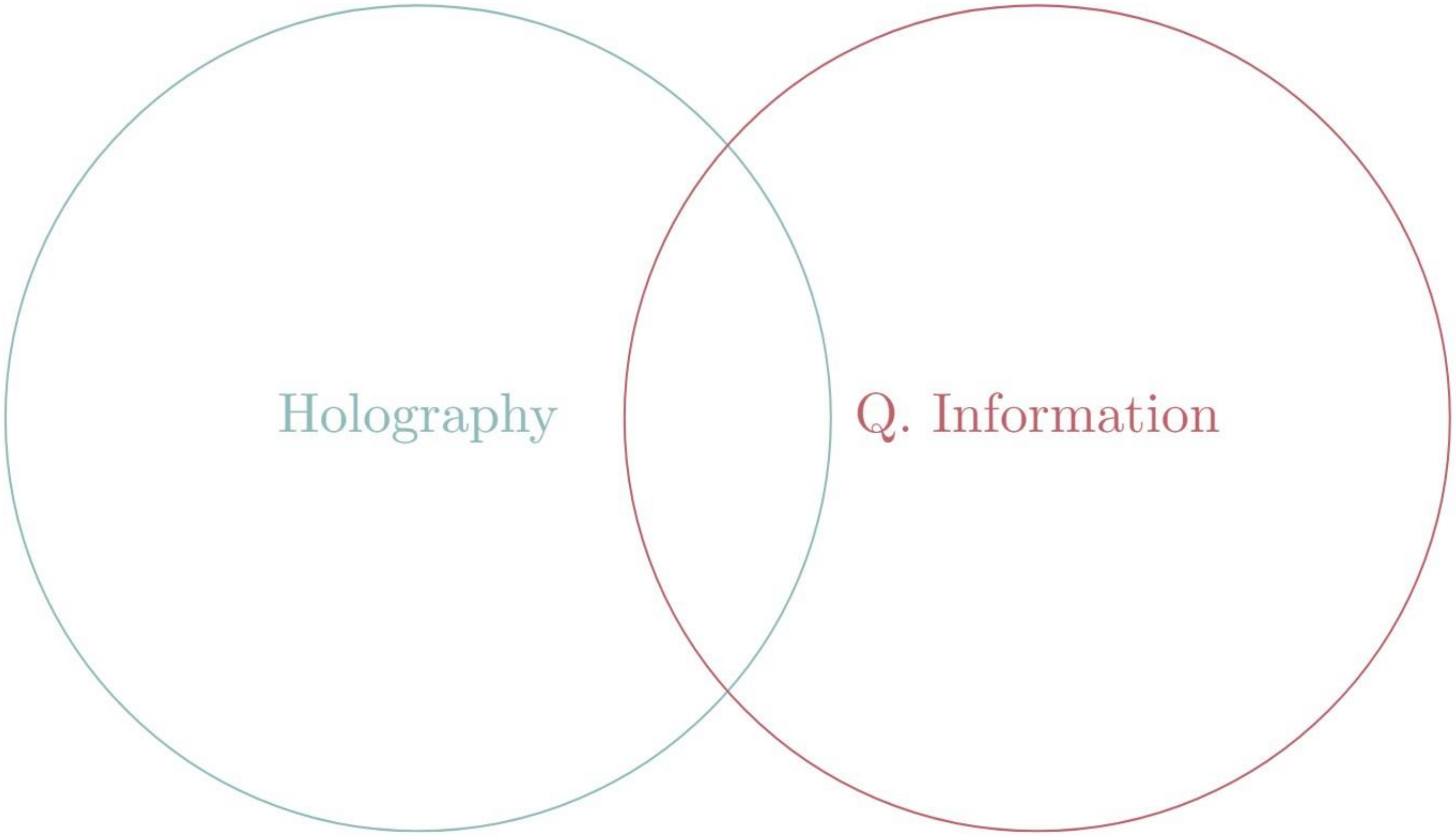

Holography
Q. Information

Quantum Error Correction

Quantum Error Correction (QEC)

QEC code: selecting an appropriate subspace of some larger Hilbert space called code subspace that has The same dimension as system

Encoding:

$$|\psi\rangle = \sum_i \lambda_i |i\rangle \in \mathcal{H}_{system} \longrightarrow |\psi\rangle_{\mathcal{C}} = \sum_i \lambda_i |i\rangle_{\mathcal{C}} \in \mathcal{H}_{\mathcal{C}}$$

Each code can correct a specific set of errors

$$\mathcal{E} = \{E_1, E_2, ..., E_n\}$$

2-step error correcting procedure:

1. Error detection
2. Recovery

Quantum Error Correction (QEC)

Example: 1 to 3 qubits encoding

$$|0\rangle \longrightarrow |0\rangle_{\mathcal{C}} = |000\rangle \qquad |1\rangle \longrightarrow |1\rangle_{\mathcal{C}} = |111\rangle$$

$$\mathcal{H}_{code} = \text{span}\{|000\rangle, |111\rangle\}$$

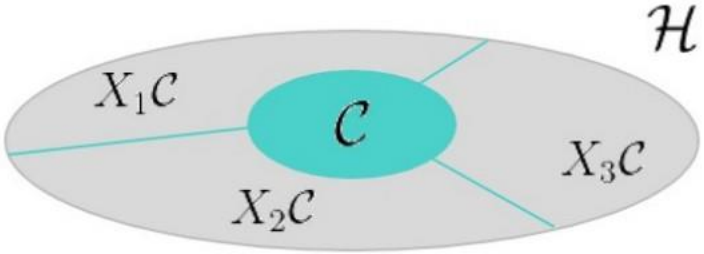
$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \quad \longrightarrow \quad |\psi\rangle_{\mathcal{C}} = a\,|000\rangle + b\,|111\rangle$$

Set of the errors: $\mathcal{E} = \{X_1, X_2, X_3\}$

Quantum Error Correction (QEC)

Example: 1 to 3 qubits encoding

$$\{|\psi\rangle_{\mathcal{C}}, X_1|\psi\rangle_{\mathcal{C}}, X_2|\psi\rangle_{\mathcal{C}}, X_3|\psi\rangle_{\mathcal{C}}\}$$



	$\|\psi\rangle_E$	$X_1\|\psi\rangle_E$	$X_2\|\psi\rangle_E$	$X_3\|\psi\rangle_E$
Z_1Z_2	1	-1	-1	1
Z_1Z_3	1	-1	1	-1

Quantum Channel

$\mathcal{E} : L(\mathcal{H}) \to L(\mathcal{H})$

Completely positive and trace preserving

Kraus Representation:

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger \qquad \sum_i A_i^\dagger A_i = I$$

Model Environment :

$$\mathcal{E}(.) = \mathrm{tr}_{en}\big(U(. \otimes \sigma_{en})U^\dagger\big)$$

General theory of QEC

Noise model: quantum channel with Kraus operators correspond with the set of errors

Error correction via Recovery channel: $\mathcal{R} \circ \mathcal{E}(\rho) = \rho \qquad \forall \rho = P\rho P$

Example: 1 to 3 qubits encoding

$$\mathcal{E}(\rho) = q_0 \rho + \sum_i q_i X_i \rho X_i \qquad \sum_i q_i = 1$$

$$\mathcal{R}(\rho) = P_0 \; \rho \; P_0 + \sum_i X_i P_i \; \rho \; P_i X_i$$

Operator Algebra QEC

In the Heisenberg Picture:

$$Tr(\rho\mathcal{E}^*(O)) = Tr(\mathcal{E}(\rho)O) \qquad \forall \rho, O.$$

The conservation of the states via the Recovery channel implies:

$$P(\mathcal{R}\circ\mathcal{E})^*(O)P = P\mathcal{E}^*\circ\mathcal{R}^*(O)P = POP, \qquad \forall O \in \mathcal{S}$$

Operator Algebra QEC!

Petz theorem

Relative Entropy:

$$S(\rho||\sigma) \;=\; \mathrm{tr}(\rho \log \rho \;-\; \rho \log \sigma)$$

Reversibility:

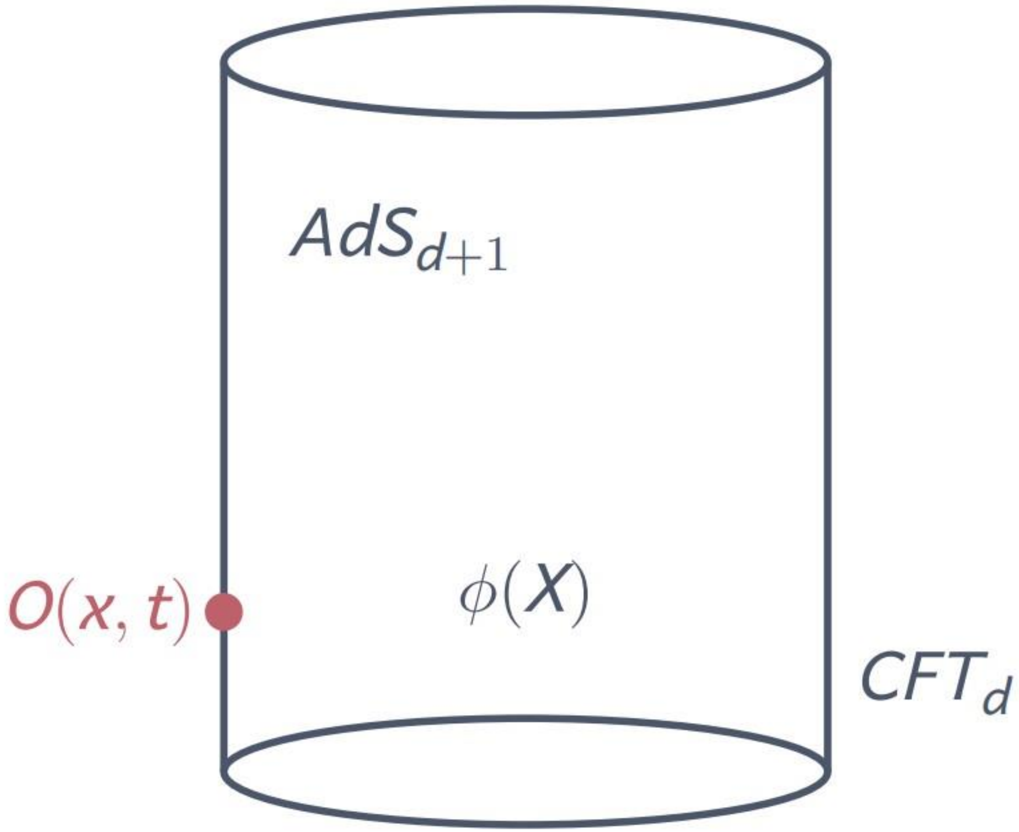
$$\exists \mathcal{R} \qquad \Longleftrightarrow \qquad S(\rho||\sigma) = S(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$$

Petz Recovery Channel:

$$\mathcal{R}(.) = \mathcal{P}_{\sigma,\mathcal{E}}(.) = \sigma^{1/2}\mathcal{E}^*\big(\mathcal{E}(\sigma)^{-1/2}(.)\mathcal{E}(\sigma)^{-1/2}\big)\sigma^{1/2}$$

AdS/CFT

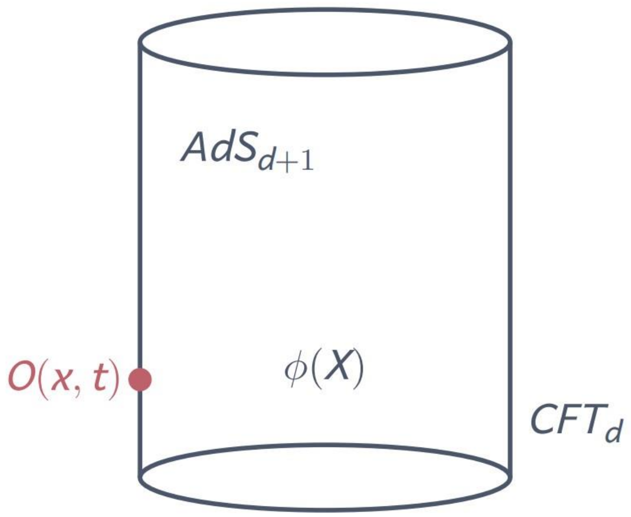
Bulk Reconstruction



$$O(t,\Omega) = lim_{\rho\to\pi/2}\frac{1}{cos^{\Delta}\rho}\phi(t,\rho,\Omega)$$

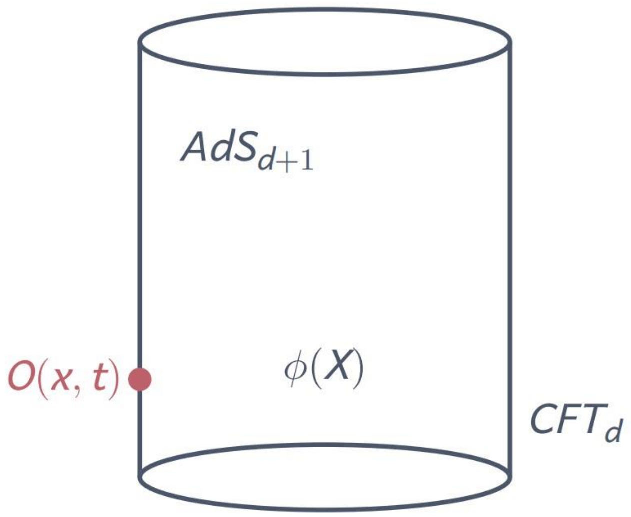
HKLL procedure

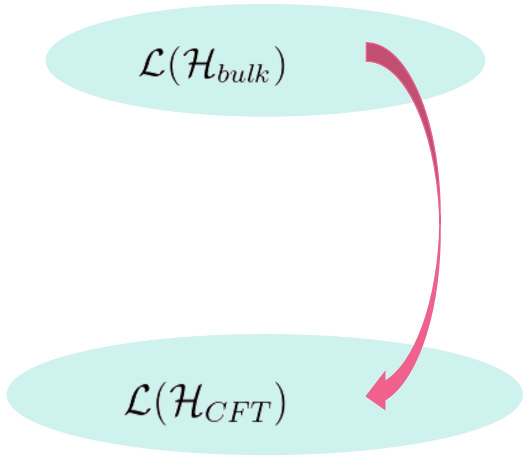
- Fields obey the bulk E.O.M
- Extrapolate Dictionary



$$\phi(X) \quad \leftrightarrow \quad \Phi_{HKLL}(X) = \int_{bdy} dtdx\, K(X|x,t)\ O(x,t)$$

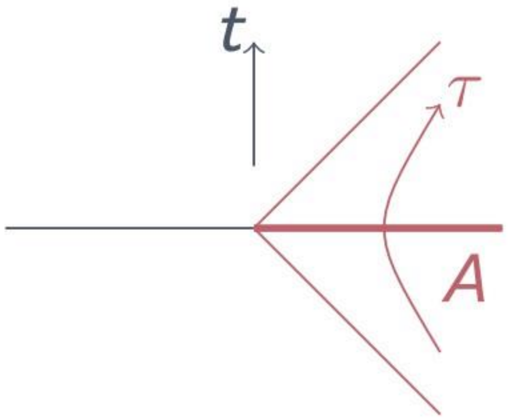
HKLL procedure

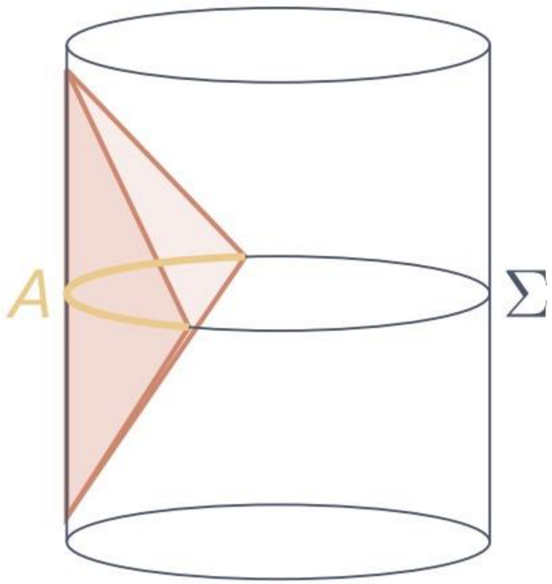




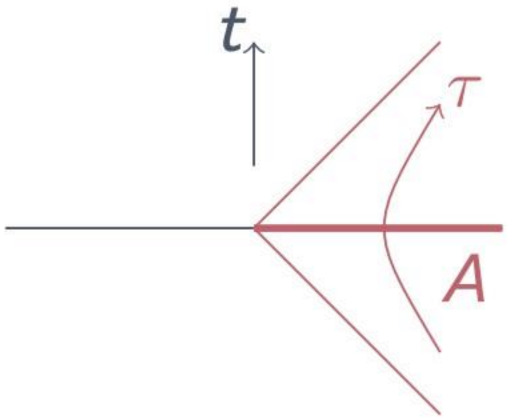
$$V_{HKLL}\ :\qquad \phi(x) \longrightarrow V_{HKLL}\phi(x)V^{\dagger}_{HKLL} = \Phi_{CFT}(x)$$

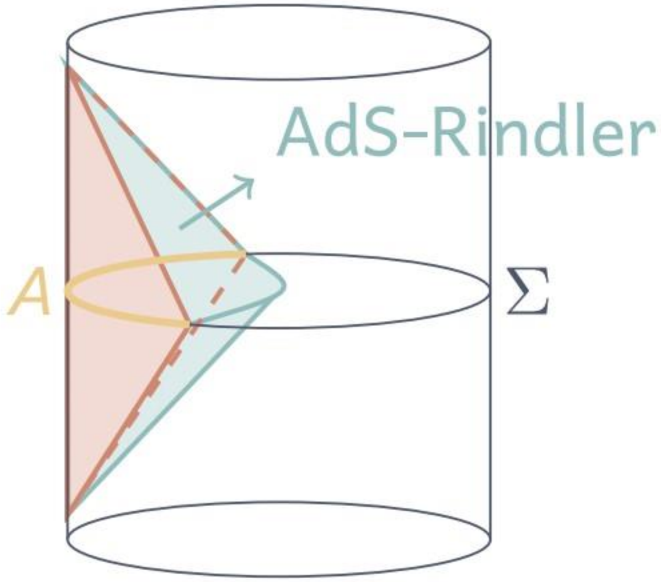
AdS-Rindler Reconstruction





AdS-Rindler Reconstruction





$$\Phi_{HKLL,Rindler}(X) = \int_{\mathcal{D}_A} d^{d-1}y\, d\tau\; K(X|y,\tau)\; O(y,\tau)$$

QEC & Holography

3 Qutrit Code!

$$|\widetilde{0}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$$
$$|\widetilde{1}\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle)$$
$$|\widetilde{2}\rangle = \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle)$$

$$|\widetilde{\psi}\rangle = U_{12}(|\psi\rangle_1 \otimes |\chi\rangle_{23})$$

$$|\chi\rangle \equiv \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

$$\begin{array}{lll} |00\rangle \to |00\rangle & |11\rangle \to |20\rangle & |22\rangle \to |10\rangle \\ |01\rangle \to |11\rangle & |12\rangle \to |01\rangle & |20\rangle \to |21\rangle \\ |02\rangle \to |22\rangle & |10\rangle \to |12\rangle & |21\rangle \to |02\rangle \end{array}$$

3 Qutrit Code!

$$|\widetilde{0}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$$
$$|\widetilde{1}\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle)$$
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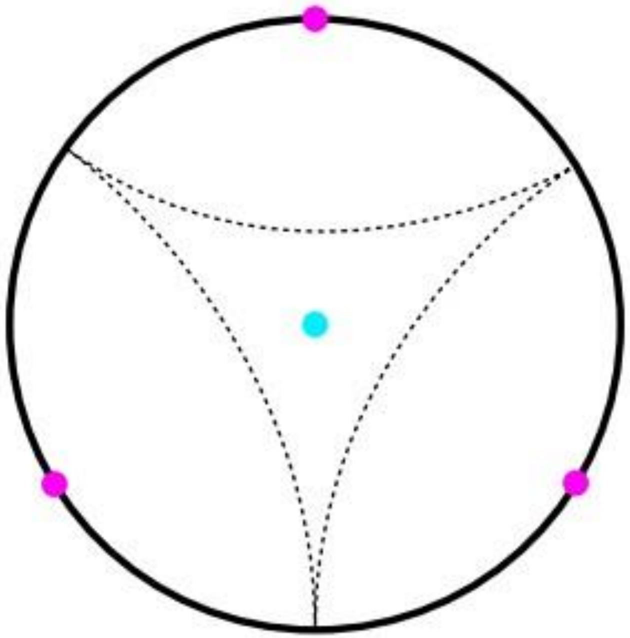


$$|\widetilde{\psi}\rangle = U_{12}\left(|\psi\rangle_1 \otimes |\chi\rangle_{23}\right)$$

$$|\widetilde{\psi}\rangle = U_{23}\left(|\psi\rangle_2 \otimes |\chi\rangle_{13}\right)$$
$$|\widetilde{\psi}\rangle = U_{31}\left(|\psi\rangle_3 \otimes |\chi\rangle_{12}\right)$$

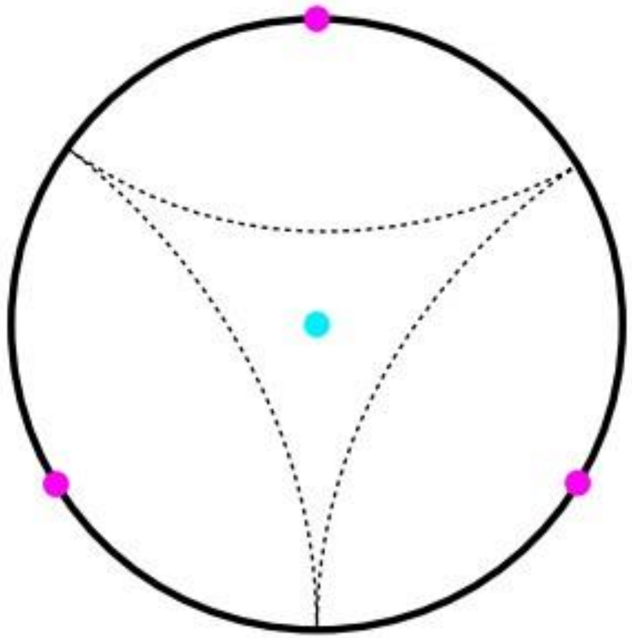
Holographic Map as a QEC code!

Logical operator:

$$O|i\rangle = \sum_j (O)_{ji}|j\rangle$$

$$\widetilde{O}|\widetilde{i}\rangle = \sum_j (O)_{ji}|\widetilde{j}\rangle$$

$$\widetilde{O}^\dagger|\widetilde{i}\rangle = \sum_j (O)^*_{ij}|\widetilde{j}\rangle$$



$$|\widetilde{\psi}\rangle = U_{12}\left(|\psi\rangle_1 \otimes |\chi\rangle_{23}\right)$$

$$O_{12} \equiv U_{12} O_1 U_{12}^\dagger$$

$$O_{12}|\widetilde{\psi}\rangle = \widetilde{O}|\widetilde{\psi}\rangle$$

$$O_{12}^\dagger|\widetilde{\psi}\rangle = \widetilde{O}^\dagger|\widetilde{\psi}\rangle$$

$$\langle\widetilde{\phi}|[\widetilde{O}, X_3]|\widetilde{\psi}\rangle = \langle\widetilde{\phi}|[O_{12}, X_3]|\widetilde{\psi}\rangle = 0$$

A theorem in OAQEC

$$\mathcal{H}_{\mathcal{C}} \subset \mathcal{H}_E \otimes \mathcal{H}_{\overline{E}}$$

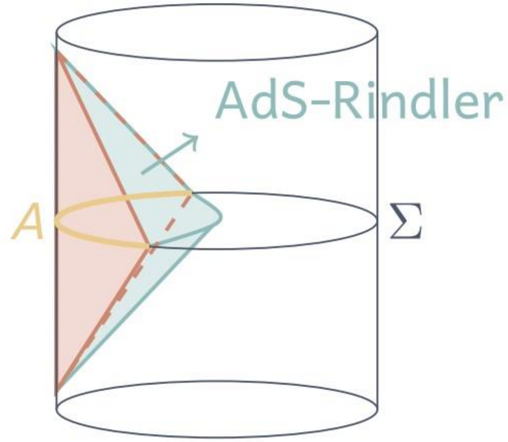
$$O|\widetilde{i}\rangle = \sum_j O_{ji}|\widetilde{j}\rangle$$

$$O^\dagger|\widetilde{i}\rangle = \sum_j O^*_{ij}|\widetilde{j}\rangle$$

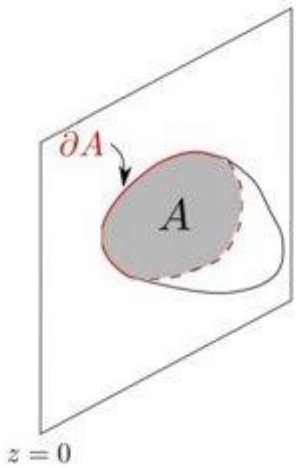
$$O_{\overline{E}}|\widetilde{\psi}\rangle = O|\widetilde{\psi}\rangle\,, \quad O^\dagger_{\overline{E}}|\widetilde{\psi}\rangle = O^\dagger|\widetilde{\psi}\rangle \quad \Longleftrightarrow \quad \langle\widetilde{i}|[O, X_E]|\widetilde{j}\rangle = 0 \qquad \forall i, j.$$

Causal wedge vs Entanglement wedge

Causal Wedge → HKLL procedure



Entanglement Wedge



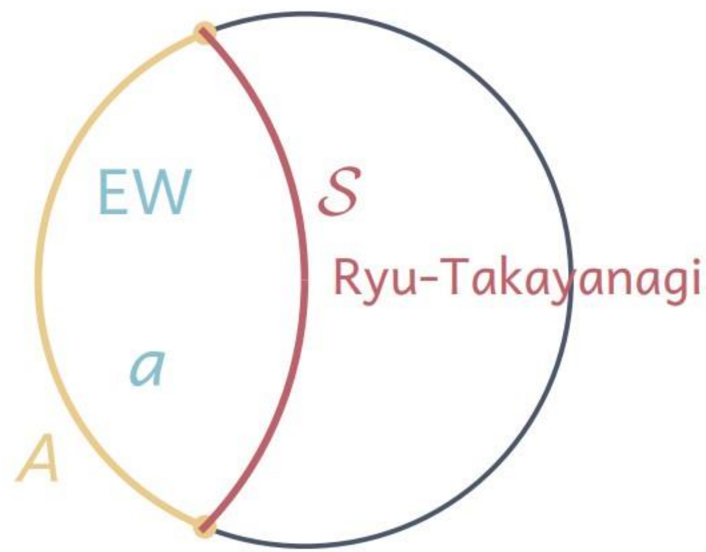
$$S_A = \frac{Area(X)}{4G_N}$$

JLMS statement

$\rho = e^K$ $\quad K$: *Modular Hamiltonian*

$$K_{bdy,A} = \frac{Area(\mathcal{S})}{4G} + K_{bulk,a} + O(1/N)$$

$$S(\rho_A|\sigma_A) = S(\rho_a|\sigma_a) + O(1/N)$$



Entanglement Wedge Reconstruction

$$\rho_{\overline{A}} \equiv \mathrm{Tr}_A|\phi\rangle\langle\phi|\,, \qquad \sigma_{\overline{A}} \equiv \mathrm{Tr}_A|\psi\rangle\langle\psi|$$

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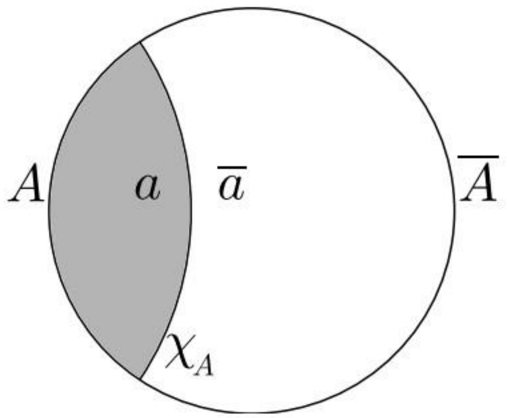
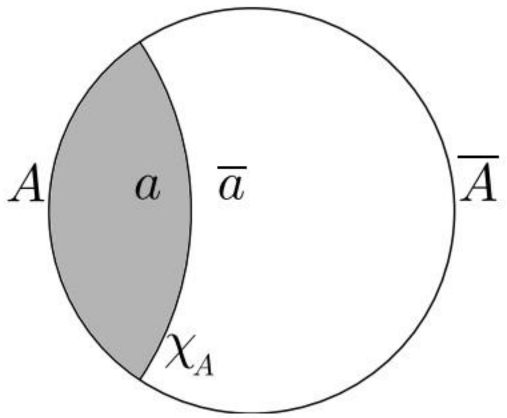


$$\mathcal{H}_{code} = \mathcal{H}_a \otimes \mathcal{H}_{\overline{a}}$$

$$\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$$

Entanglement Wedge Reconstruction

JLMS statement: $\rho_{\overline{a}} = \sigma_{\overline{a}} \quad \Rightarrow \quad \rho_{\overline{A}} = \sigma_{\overline{A}}$



O acts only on H_a:

$$|\psi\rangle \equiv e^{i\lambda O}|\phi\rangle$$

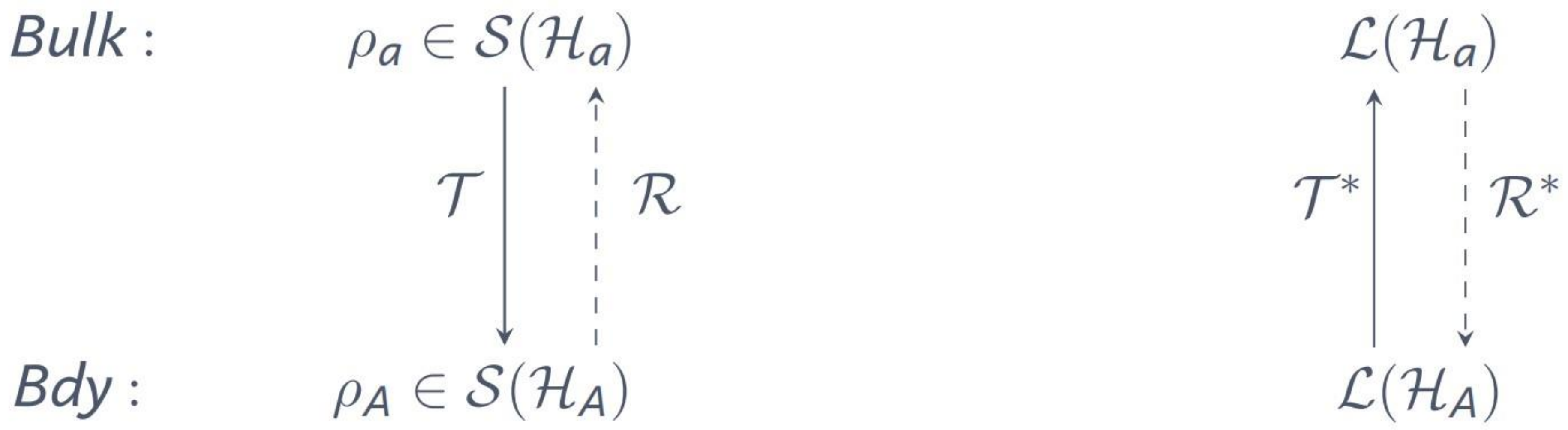
→ $\rho_{\overline{A}} = \sigma_{\overline{A}} \quad \Rightarrow \quad \langle\psi|X_{\overline{A}}|\psi\rangle = \langle\phi|X_{\overline{A}}|\phi\rangle$

→ $\langle\phi|e^{-i\lambda O}X_{\overline{A}}e^{i\lambda O}|\phi\rangle - \langle\phi|X_{\overline{A}}|\phi\rangle = 0$

$$\mathcal{H}_{code} = \mathcal{H}_a \otimes \mathcal{H}_{\overline{a}}$$

$$\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$$

$\langle\phi|[O, X_{\overline{A}}]|\phi\rangle = 0$ → $O_A|\phi\rangle = O|\phi\rangle\,, \quad O_A^\dagger|\phi\rangle = O^\dagger|\phi\rangle$ EWR!



JLMS statement:

$$S(\mathcal{T}(\rho_a)|\mathcal{T}(\sigma_a)) = S(\rho_a|\sigma_a)$$

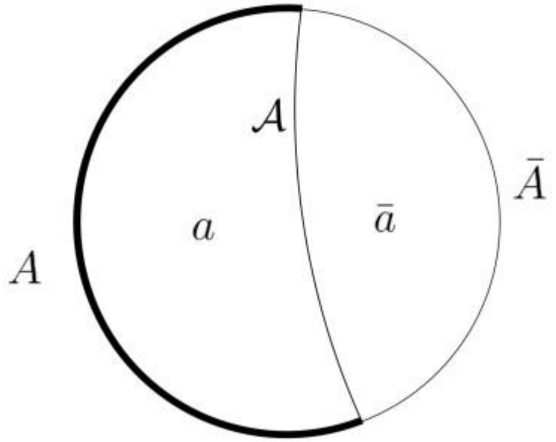
Recovery Map:

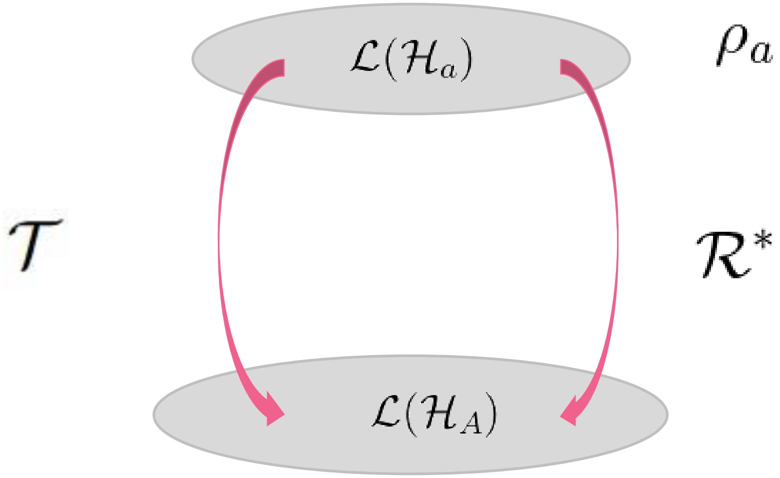
$$\exists \mathcal{R}, \qquad \mathcal{R}.\mathcal{T}(\rho_a) = \rho_a$$

$\mathcal{T}^*$: Dual Map

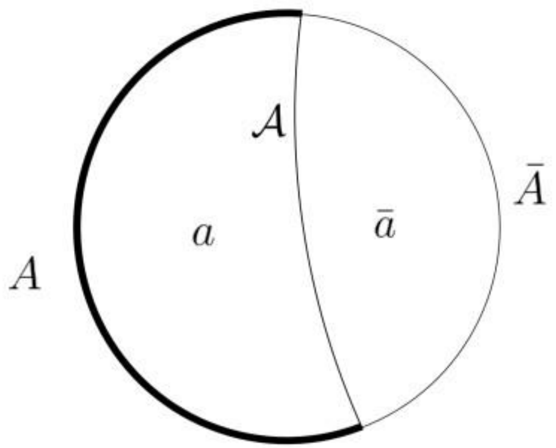
$$Tr[\rho \mathcal{T}^*(O)] = Tr[\mathcal{T}(\rho) O]$$

EWR & Recovery Channel

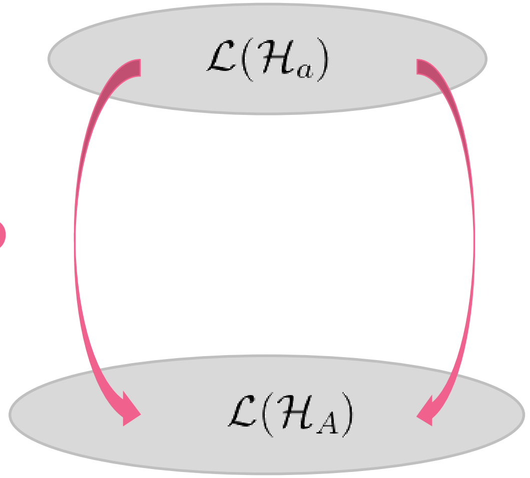




EWR & Recovery Channel



$\mathcal{T}$?



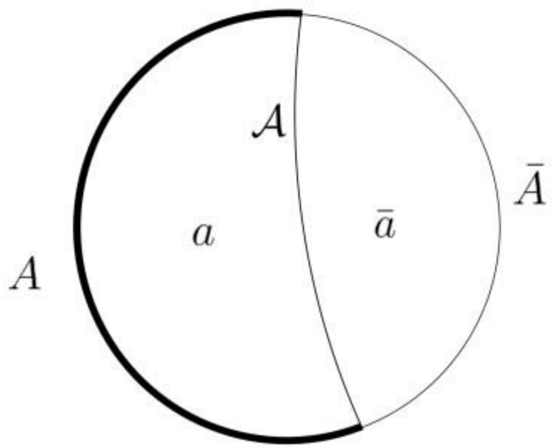
ρ_a

Petz Recovery Channel

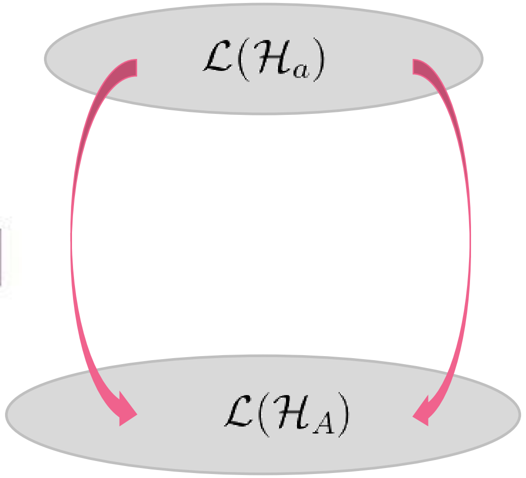
$$\mathcal{R}^*(.) = \rho_A^{-1/2}\mathcal{T}(\rho_a^{1/2}(.)\rho_a^{1/2})\rho_A^{-1/2}$$

$\rho_A = \mathcal{T}(\rho_a)$

EWR & Recovery Channel



$$\mathcal{T}(.) = Tr_{\bar{A}}[V_{HKLL}(. \otimes \rho_{\bar{a}})V^{\dagger}_{HKLL}]$$



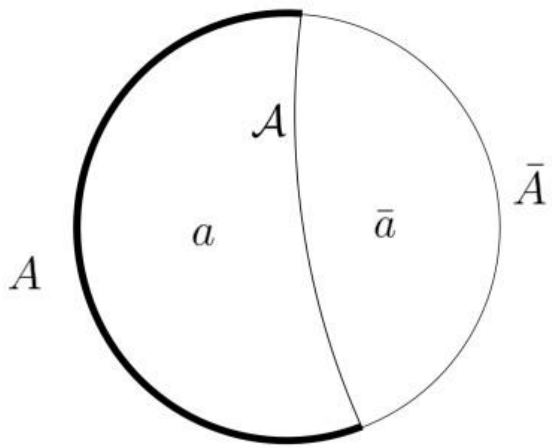
ρ_a

Petz Recovery Channel

$$\mathcal{R}^*(.) = \rho_A^{-1/2}\mathcal{T}(\rho_a^{1/2}(.)\rho_a^{1/2})\rho_A^{-1/2}$$

$$\rho_A = \mathcal{T}(\rho_a)$$

Petz Map



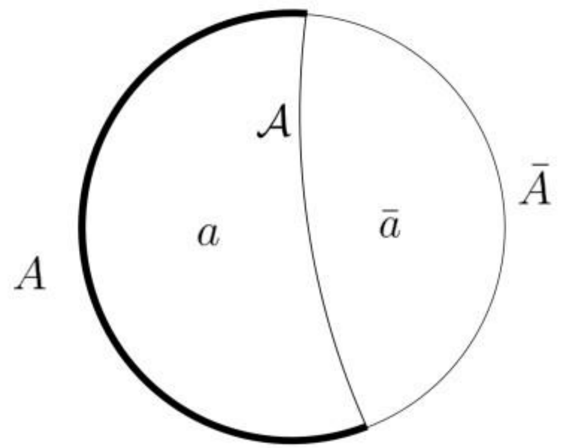
$$\Phi_{CFT} = \mathcal{R}^*(\phi_{bulk})$$
$$= \rho_A^{-1/2} Tr_{\bar{A}}[V_{HKLL}(\rho_a^{1/2}\phi_{bulk}\rho_a^{1/2})V_{HKLL}^\dagger]\rho_A^{-1/2}$$

1. $\rho_a, \rho_{\bar{a}}$: maximally mixed states $\quad \Longrightarrow \quad \rho_a \otimes \rho_{\bar{a}} = \frac{I_{bulk}}{d_{code}} = \tau_{bulk}$

2. conditions on expectation values: $\langle \phi_a \rangle_{\rho_{bulk}} = \langle \Phi_{HKLL} \rangle_{\rho_{CFT}}$

$$V_{HKLL}\phi_a V_{HKLL}^\dagger = P_{code}\Phi_{HKLL}P_{code}$$

Petz Map

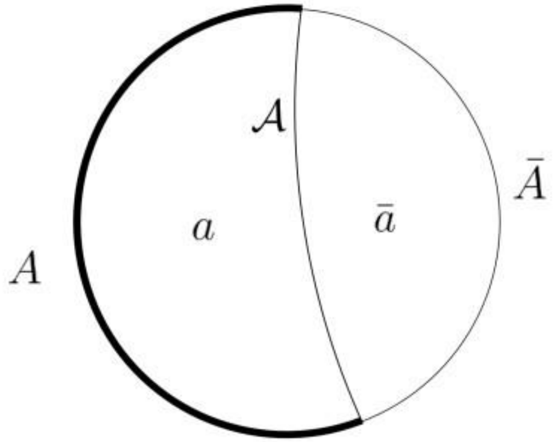


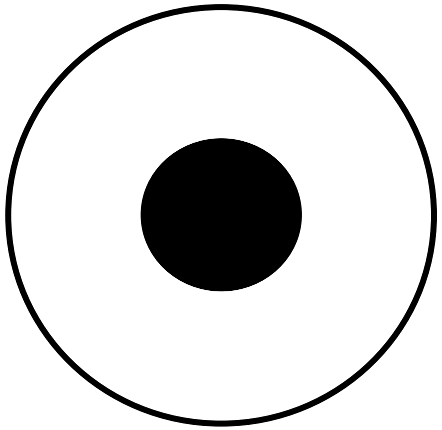
$$O_A = \frac{1}{d_{code}} \tau_A^{-1/2} \mathrm{tr}_{\bar{A}} \left(V_{HKLL} (\phi_a) V_{HKLL}^{\dagger} \right) \tau_A^{-1/2}$$

Petz Map !

$$\tau_A = \frac{1}{d_{code}} \mathrm{tr}_{\bar{A}} P_{code}$$

Petz Map Reconstruction





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[N. Vardian]

$$\phi_a(X) \quad \Longrightarrow \quad O_A^{Petz} = \int_A dx_A \int_{-\infty}^{\infty} ds\ K_{Petz}(X|x_A, s)\rho_A^{is} O(x_A)\rho_A^{-is}$$

Thank You !