Holographic Entanglement of Purification near a Critical Point

B.Amrahi*, M.Ali-Akbari†, M.Asadi‡

* b_amrahi@sbu.ac.ir

† m_aliakbari@sbu.ac.ir

‡ m_asadi@ipm.ir

*,† Shahid Beheshti University

‡ Institute for Research in Fundamental Sciences (IPM)

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Outline

- AdS/CFT correspondence

  $\mathcal{N} = 4$ $SU(N)$ SYM theory is equivalent to type IIB string theory in $AdS_5 \times S_5$

- gauge-gravity duality

  any strongly coupled CFT is equivalent to a classical gravity in one higher dimension

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Definition

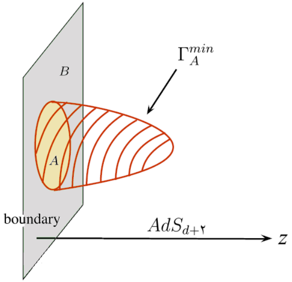
When the total Hilbert space $\mathcal{H}_{tot}$ is decomposed into a direct product $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$, we define the reduced density matrix ρ_A by $\rho_A = Tr_{A^c}\rho_{tot}$, where ρ_{tot} is the total density matrix. The entanglement entropy $S(\rho_A)$ for the subsystem A is defined by

$$S(\rho_A) = -Tr\rho_A \log \rho_A \tag{1}$$

$I(A:B)$ is mutual information between subsystems A and B and defined by this equation

$$I(A:B) = S_A + S_B - S_{A\cup B} \tag{2}$$

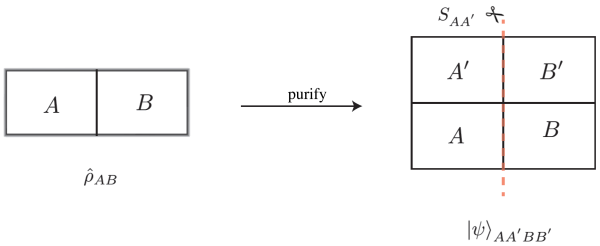
Holographic Entanglement Entropy



The holographic entanglement entropy is given by

$$S(\rho_A) = \frac{Area(\Gamma_A^{min})}{4G_N} \quad (3)$$

Purification



ρ_{AB} is a density matrix on a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$. We can purify this mixed state by enlarging its Hilbert space. $|\psi\rangle_{AA'BB'} \in \mathcal{H}_{AA'} \otimes \mathcal{H}_{BB'}$ is a purification of ρ_{AB}, so that $Tr_{A'B'}|\psi\rangle_{AA'BB'}\langle\psi| = \rho_{AB}$. There exists infinite ways to purify ρ_{AB}.

Entanglement of purification

EoP is given by

$$E_p(\rho_{AB}) = \min_{\rho_{AB}=Tr_{A'B'}(|\psi\rangle_{AA'BB'}\langle\psi|)} S(\rho_{AA'}) \tag{4}$$

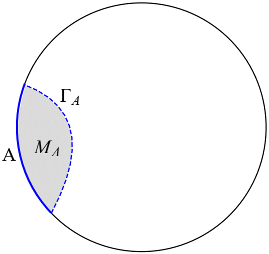
where $S_{AA\prime}$ is the entanglement between $AA' = A \cup A'$ and $BB' = B \cup B'$.

$$S_{AA'}(\rho_{AA'}) = -Tr(\rho_{AA'}\log(\rho_{AA'})) \tag{5}$$

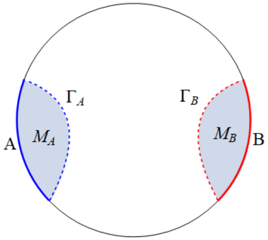
and $\rho_{AA'} = Tr_{BB'}(|\psi\rangle_{AA'BB'}\langle\psi|)$

Entanglement wedge

The entanglement wedge of subsystem A is defined as the domain of dependence of M_A. M_A is a surface that is surrounded between A and Γ_A. In fact M_A is a time slice of entanglement wedge of subsystem A. The entanglement wedge of subsystem A is dual with density matrix ρ_A.

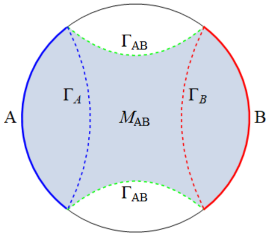


Entanglement wedge



$\rho_{AB} = \rho_A \otimes \rho_B$
no correlation ($S_{AB} = S_A + S_B$)

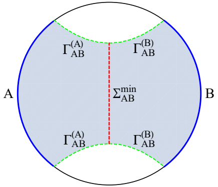
$M_{AB} = M_A \cup M_B$



$\rho_{AB} \neq \rho_A \otimes \rho_B$
correlated ($S_{AB} > S_A + S_B$)

$M_{AB} \neq M_A \cup M_B$

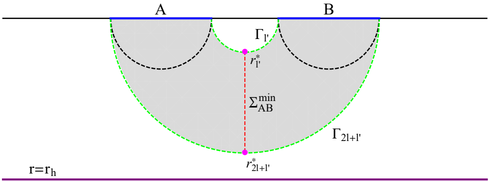
Entanglement wedge cross section(E_w)



$$E_w = \frac{Area(\Sigma_{AB}^{min})}{4G_N} \tag{6}$$

E_w is a measure of correlation between A and B and has all properties of EoP. So it is conjectured that the EoP is holographically dual to entanglement wedge cross section of ρ_{AB}. As a result we have

$$E_p(\rho_{AB}) = E_w(\rho_{AB}) \tag{7}$$



We consider symmetric case where the length of the both disjoint subsystems is equal and consider a general metric

$$ds^2 = f_1(r)dt^2 + f_2(r)dr^2 + f_3(r)dx_i^2 \quad , \quad i = 1, 2, ..., d \tag{8}$$

$r \to \infty$ is the AdS boundary.

$$Ew(A:B) = \frac{L^{d-1}}{4G_N} \int_{r^*_{2l+l'}}^{r^*_{l'}} dr \sqrt{f_2 f_3^{d-1}} \tag{9}$$

$$\frac{l'}{2} = \int_{r^*_{l'}}^{\infty} dr \sqrt{\frac{f_2 f_{3*}^d}{f_3(f_3^d - f_{3*}^d)}} \tag{10}$$

background with a critical point

$$ds^2 = \exp^{2A(R)}(-h(r)dt^2 + d\vec{x}^2) + \frac{\exp^{2B(r)}}{h(r)}dr^2, \quad (11)$$

where in this case $d = 3$ and

$$A(r) = \ln\left(r(1+\frac{Q^2}{r^2})^{\frac{1}{6}}\right), \quad B(r) = -\ln\left(r(1+\frac{Q^2}{r^2})^{\frac{1}{3}}\right), \quad h(r) = 1 - \frac{M^2}{r^2(r^2+Q^2)}.$$

$$h(r_h) = 0 \Longrightarrow r_h = \sqrt{\frac{\sqrt{Q^4+4M^2} - Q^2}{2}}. \quad (12)$$

$$T = \frac{2r_h^2 + Q^2}{2\pi\sqrt{Q^2+r_h^2}}, \quad \mu = \frac{Qr_h}{\sqrt{Q^2+r_h^2}}. \quad (13)$$

It was shown that there is a critical point at $\frac{\mu}{T} = (\frac{\mu}{T})_* = \frac{\pi}{\sqrt{2}}$ $(\frac{Q}{r_h} = \sqrt{2})$ and the solutions are thermodynamically stable for $\frac{Q}{r_h} < \sqrt{2}$.

$$E_p \equiv \frac{4G_N}{L^2}E_w = \frac{L^2}{4G_N}\int_{r_{l\prime}^*}^{r_{2l+l\prime}^*} dr \frac{r}{1-\frac{M^2}{r^2(r^2+Q^2)}}. \quad (14)$$

background without a critical point

we consider RN-AdS$_{d+2}$ metric in the AdS radius unit

$$ds^2 = -r^2 f(r) dt^2 + \frac{1}{r^2 f(r)} dr^2 + r^2 d\vec{x}^2, \qquad f(r) = 1 - \frac{M}{r^{d+1}} + \frac{Q^2}{r^{2d}}. \tag{15}$$

$$M = r_h^{d+1} + \frac{Q^2}{r_h^{d-1}}. \tag{16}$$

$$T = \frac{r_h}{4\pi}\left((d+1) - (d-1)\frac{Q^2}{r_h^{2d}}\right), \qquad \mu = \sqrt{\frac{d}{2(d-1)}}\frac{Q}{r_h^{d-1}}. \tag{17}$$

$$\frac{\mu}{T} = \frac{1}{\sqrt{2(d-1)}} \frac{4\pi\sqrt{d} Q r_h^d}{(d+1) r_h^{2d} - (d-1) Q^2}. \tag{18}$$

$$E_p \equiv \frac{4 G_N}{L^2} E_w = \int_{r^*_{2l+l'}}^{r^*_{l'}} dr \frac{r^{d-2}}{\sqrt{1 - \frac{M}{r^{d+1}} + \frac{Q^2}{r^{2d}}}}. \tag{19}$$

We have checked the inequality between EoP and the mutual information, i.e. $\frac{I}{2} \leq E_p$

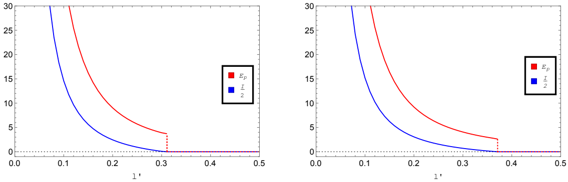


Figure: The EoP and $I/2$ with respect to l' for $l = 0.5$ (left) and $l = 0.8$ (right).

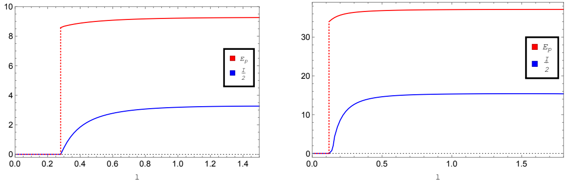
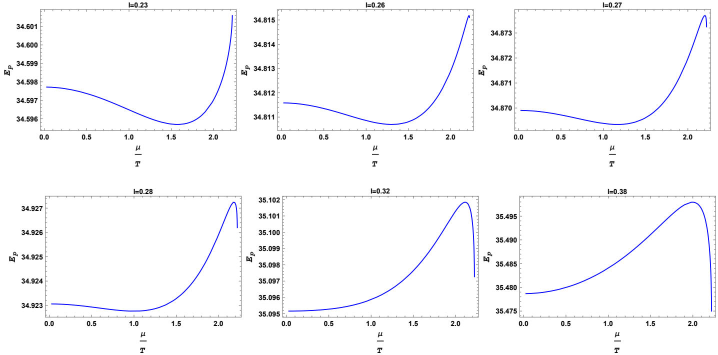


Figure: E_p and $I/2$ with respect to l for $l' = 0.1$ (left) and $l' = 0.2$ (right).

The EoP with respect to $\frac{\mu}{T}$ for $l' = 0.1$ and differenet values of l. T is fixed.



- The EoP is not a monotonic function of scale $\frac{\mu}{T}$.
- The non-trivial behavior of EoP depends on the values of l and l'.
- There are two or three different configurations, labeled by various values of $\frac{\mu}{T}$s, with the same EoP.

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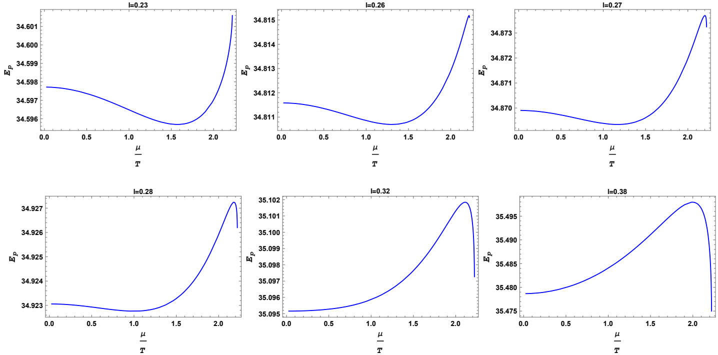

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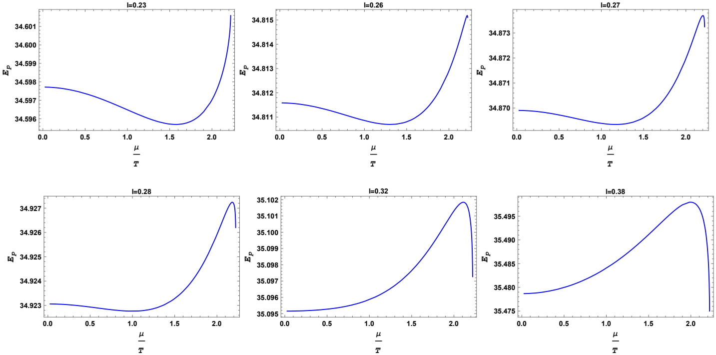

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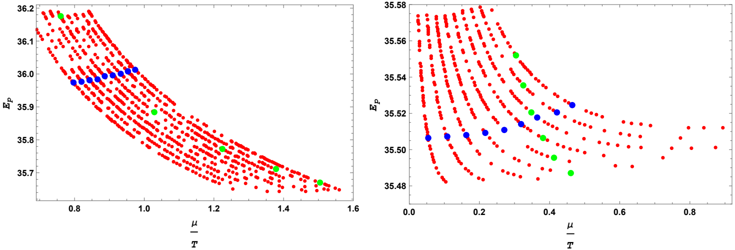
The EoP with respect to $\frac{\mu}{T}$ for $l' = 0.1$ and $l = 0.5$ in the field theory with (left) and without (right) critical point. The green points show the configuration at fixed μ and blue points show the configuration at fixed T.



- There are many points with different values of $\frac{\mu}{T}$ which have the same value of EoP.
- The EoP or the correlation between the subsystems increases by raising both temperature and/or chemical potential.
- The EoP, as a function of $\frac{\mu}{T}$, is not a good observable for distinguishing a critical point between the holographic field theories.

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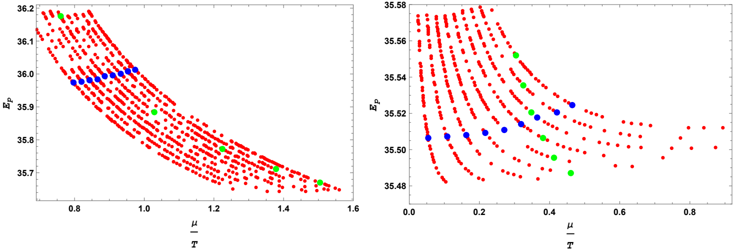

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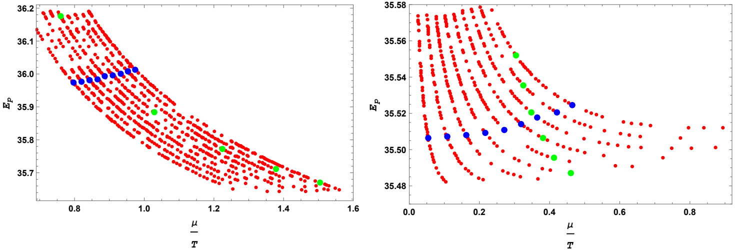

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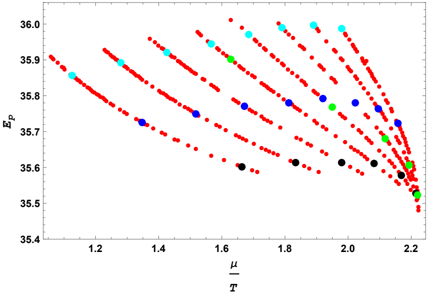
The EoP in terms of $\frac{\mu}{T}$ near the critical point. The cyan, blue and black points show $T = 0.995$, 0.805 and 0.61, respectively. The green points shows $\mu = 1.53$.



- All curves, both fixed temperature and chemical potential curves, converge at $\frac{\mu}{T} = (\frac{\mu}{T})_*$.
- Near the critical point we have $\frac{dE_P}{d(\frac{\mu}{T})} \propto (\frac{\mu}{T} - (\frac{\mu}{T})_*)^{-\theta}$ and therefore close to this point the number θ, called critical exponent, describes the variation of the EoP with respect to $\frac{\mu}{T}$

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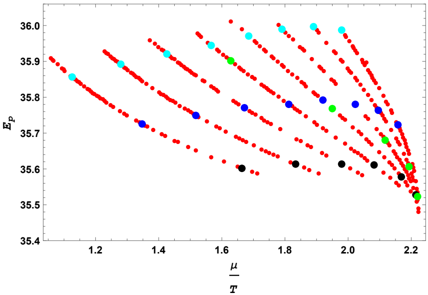

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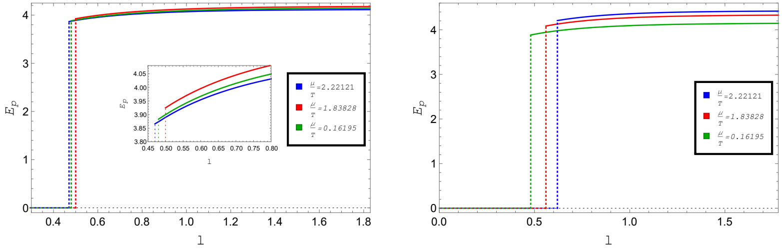
The EoP with respect to l for three different values of $\frac{\mu}{T}$ and $l' = 0.3$. The left (right) panel has been plotted for the field theories dual to (11) ((15) with $d = 3$).



- In the right panel, the EoP and $\frac{\mu}{T}$ increase together. However, in the left panel one can see that the EoP has no general behavior and near the critical point it decreases or increases.
- For large enough l, the EoP does not change substantially with distance l for given $\frac{\mu}{T}$.
- the EoP, as a function of $\frac{\mu}{T}$ and l, distinguishes which theory has a critical point.

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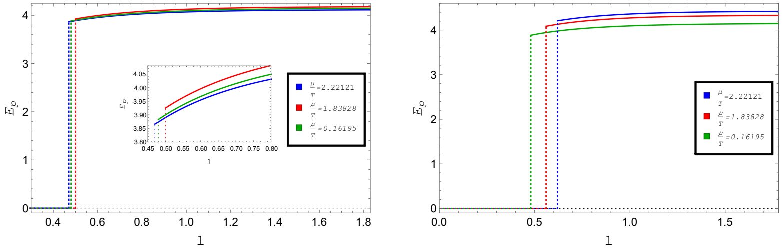

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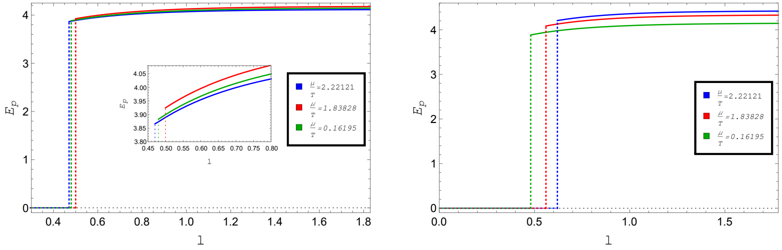

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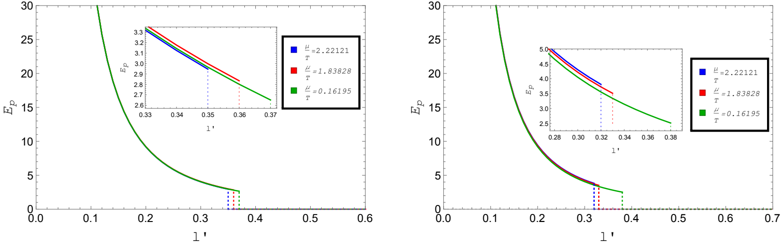

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- The EoP decreases substantially as the distance between subsystems becomes larger.

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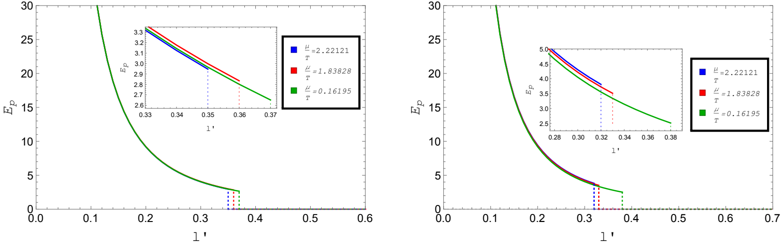

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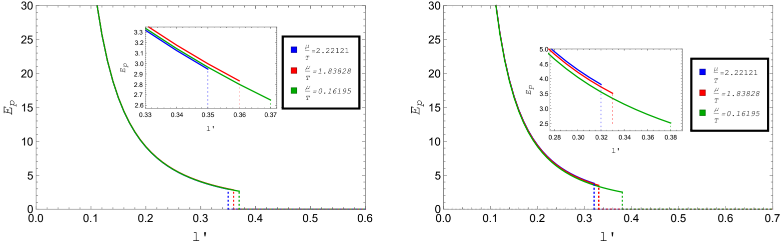

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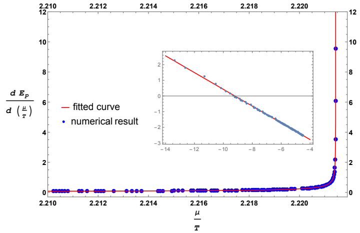
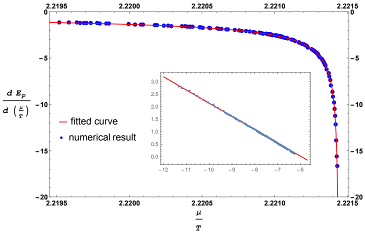
The slope of E_p with respect to $\frac{\mu}{T}$. The left diagram has been plot for $l = 0.2$ and $l' = 0.1$ and right diagram has been plot for $l = 0.4$ and $l' = 0.2$. For the left (right) figure θ will be obtained 0.534 (0.526).





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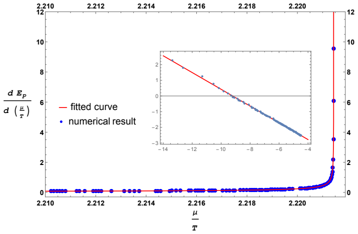

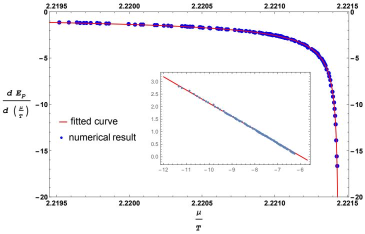

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Main References

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Tanx for your attention!