Holographic Superconductor

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The plan on the lecture.

- Brief introduction to AdS/CFT correspondence
- Brief review of superconductivity
- Holographic superconductor

Brief introduction to AdS/CFT correspondence

Basically AdS/CFT correspondence is a duality or a relation between two theories one with a gravity and the other without gravity.

The gravitational theory is usually defined in higher dimension.

Well developed case is the one where the gravity is defined on an AdS geometry where the dual theory is a CFT living in the conformal boundary of AdS space.

Classical gravity on AdS_{d+1} background is dual to d-dimensional "Large N" strongly coupled field theory on its boundary.

Large N $\longrightarrow$ when we have gauge theory, otherwise it is not clear what large N means?

It is important to note that:having AdS geometry $\longrightarrow$ having CFT.

More generally: gravitational theory on an asymptotically locally AdS geometry $\longrightarrow$ quantum field theory with UV fixed point where the theory becomes conformal.

AdS_{d+1} metric in Poincare coordinates

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2$$

AdS_{d+1} metric in global coordinates

$$ds^2 = -(1 + \frac{r^2}{R^2})dt^2 + \frac{dr^2}{1 + \frac{R^2}{r^2}} + r^2 d\Omega_{d-1}^2$$

Here boundary is at $r \to \infty$

Let's define $\rho = \frac{R^2}{r^2}$ then the asymptotically locally AdS may be given by

$$ds^2 = R^2 \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(x,\rho) dx^\mu dx^\nu$$

where

$$g_{\mu\nu}(x,\rho) = g_{\mu\nu}(x)^{(0)} + \rho g_{\mu\nu}^{(2)}(x) + \cdots + \rho^{d/2}(g_{\mu\nu}^{(d)}(x) + \log \rho \tilde{g}_{\mu\nu}^{(d)}) + \cdots$$

It may or may not have a log term!

if $g_{\mu\nu}^{(0)} = \eta_{\mu\nu}$, and all others are zero, we have AdS.

There is one to one correspondence between objects in CFT and those in the gravitational theory on AdS space.

Consider an operator in CFT with conformal dimension Δ and p indices:

$$\mathcal{O}_{\dots}$$

Let's deform the CFT by this operator as follows

$$\int d^dx \, \mathcal{L}_{\text{CFT}} + \int d^dx \, \varphi(x)\mathcal{O}(x)$$

I have dropped the indices!

There is a dual field in the bulk gravitational theory such that

$$\lim_{\rho \to 0} \Phi(x, \rho) \sim \rho^{d-\Delta-p}\varphi(x)$$

Moreover assume that $S[\Phi]$ is the classical action of Φ in the bulk. Then

$$\left\langle exp\left[\int d^d x \; \varphi(x)\mathcal{O}(x)\right]\right\rangle_{\text{CFT}} = Z_{\text{AdS}}\left[\Phi(\rho,x)\middle|\Phi(\rho\to 0)\sim\rho^{d-\Delta-p}\varphi(x)\right]$$

In the saddle point approximation

$$\left\langle exp\left[\int d^d x \; \varphi(x)\mathcal{O}(x)\right]\right\rangle_{\text{CFT}} = e^{S[\Phi(\rho,x)_{cl}|\Phi(\rho\to 0)\sim\rho^{d-\Delta-p}\varphi(x)]}$$

Therefore one can find n-point functions of the CFT using classical gravity!

Consider a generic field $\Phi(x,\rho)$ as before. Near the boundary it has an asymptotic expansion of the form

$$\Phi(x,\rho) \sim \rho^{(d-\Delta-p)/2}\left[\varphi(x)+\rho\varphi^{(1)}(x)+\cdots+\rho^{(2\Delta-d+p)/2}\left(\phi(x)+\log\rho\tilde{\phi}(x)\right)+\cdots\right]$$

There could also be a log term!

Given φ as a free parameter and using the equations of motion all other parameters φ_i can be fixed as a function of φ, except ϕ.

To find ϕ as a function of φ one needs the IR information too (to solve the equations of motion).

Having obtained the most general asymptotic solution of the field equations, we now proceed to compute the on-shell value of the action.

To regularize the on-shell action we restrict the range of the integration, $\rho \geq \epsilon$.

The on-shell action takes the form

$$S_{\text{reg}}[\varphi, \epsilon] = \int d^d x \sqrt{g_{(0)}} \left[\epsilon^{-\nu} a_{(0)} + \epsilon^{-\epsilon+1} a_{(2)} + \cdots + \log \epsilon a_{(2\nu)} + \mathcal{O}(1) \right]$$

where ν is a positive number that only depends on the scale dimension of the dual operator and $a_{(2k)}$ are local functions of the source(s) φ.

The counterterm action is defined as

$$S_{\text{ct}}[\Phi(x,\epsilon);\epsilon] = -\text{divergent terms of } S_{\text{reg}}[\varphi;\epsilon]$$

where divergent terms are expressed in terms of the fields $\Phi(x,\epsilon)$ at the regulated surface $\rho = \epsilon$.

The renormalized action is

$$S_{\text{ren}}[\varphi] = \lim_{\epsilon \to 0} S_{\text{sub}}[\Phi;\epsilon]$$

where subtracted action at the cutoff is

$$S_{\text{sub}}[\Phi(x,\epsilon);\epsilon] = S_{\text{reg}}[\varphi;\epsilon] + S_{\text{ct}}[\Phi(x,\epsilon);\epsilon]$$

The 1-point function of the operator $\mathcal{O}$ in the presence of sources is defined as

$$\langle \mathcal{O} \rangle = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta \varphi}$$

Explicit evaluation of the above expression leads to

$$\langle \mathcal{O} \rangle \sim \phi[\varphi(x)] + \text{local terms}$$

Moreover we get

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \frac{\delta^{n-1} \phi[\varphi(x)]}{\delta\varphi(x_1) \cdots \delta\varphi(x_{n-1})} |_{\varphi=0}$$

Therefore for a field with the asymptotic expansion

$$\Phi(\rho, x) \sim \rho^{(d-\Delta-p)/2}\varphi(x) + \rho^{\Delta}\phi(x)$$

φ is source and ϕ is response of the dual operator $\mathcal{O}$ with conformal dimension Δ.

From gravity point of view Δ is a function of dimension, mass,....

For example for a massive p-form one has

$$\Delta = \frac{d}{2} + \sqrt{\frac{(d-2p)^2}{4} + m^2R^2}$$

Brief review of superconductivity

Superconductivity is an electrical resistance of exactly zero which occurs in certain materials below a characteristic temperature known as critical temperature T_c.

These material are called superconductor.

It might be compared with prefect conductor which is an electrical conductor with no resistivity. it also occurs when we decrease temperature.

It is important to note that superconductivity is a quantum mechanical phenomenon. It is also characterized by a phenomenon called the Meissner effect.

The ejection of any sufficiently weak magnetic field from the interior of the superconductor as it transitions into the superconducting state.

The occurrence of the Meissner effect indicates that superconductivity cannot be understood simply as the idealization of perfect conductivity in classical physics.

A phenomenological description of superconductivity was first given by London brothers (Fritz and Heinz London in 1935) with simple equation

$$\vec{J} = -\frac{ne^2}{mc}\vec{A}$$

Here $\vec{J}$ is the superconducting current, $\vec{A}$ the vector potential, e is the charge of an electron, m, is electron mass, and n, is a phenomenological constant associated with a number density of superconducting carriers.

In terms of the electric and magnetic fields E and B one has

$$\frac{\partial \vec{J}}{\partial t} = -\frac{ne^2}{mc}\vec{E}, \qquad \nabla \times \vec{J} = -\frac{ne^2}{mc}\vec{B}$$

Known as London's equations.

If the second of London's equations is manipulated by applying Ampere's law one finds

$$\nabla^2 \vec{B} = \frac{1}{\lambda^2}\vec{B}, \qquad \lambda^2 = \frac{mc^2}{4\pi ne^2}$$

Where λ is London penetration depth which is a a characteristic length over which external magnetic fields are exponentially suppressed.

In 1950, Landau and Ginzburg described superconductivity in terms of a second order phase transition whose order parameter is a complex scalar field ϕ. (The density of superconducting electrons is $n = |\phi|^2$).

The contribution of ϕ to the free energy is

$$F = \alpha(T - T_c)|\phi|^2 + \beta|\phi|^4 + \cdots$$

where α and β are positive constants and the dots denote gradient terms and higher powers of ϕ

For $T > T_c$ the minimum of the free energy is at $\phi = 0$, while for $T < T_c$ the minimum is at a nonzero value of ϕ. This is just like the Higgs mechanism in particle physics, and is associated with breaking a $U(1)$ symmetry.

The London equation follows from this spontaneous symmetry breaking

A more complete theory of superconductivity was given by Bardeen, Cooper and Schrieffer in 1957 and is known as BCS theory.

They showed that interactions with phonons can cause pairs of elections with opposite spin to bind and form a charged boson called a Cooper pair.

Below a critical temperature T_c, there is a second order phase transition and the Cooper pair, being bosons, condenses.

The DC conductivity becomes infinite producing a superconductor.

It was thought that the highest T_c for a BCS superconductor was around 30 K.

The highest T_c known today (at atmospheric pressure) is $T_c = 134$ K.

There is evidence that electron pairs still form in these high T_c materials, but the pairing mechanism is not well understood.

Unlike BCS theory,it involves strong coupling.

Gauge/gravity duality is an new tool to study strongly coupled field theories.

Holographic superconductor

How to construct a holographic model for superconductor. This means we want to have gravity dual which exhibits certain features of superconductivity.

We will consider the minimal ingredients we need.

We want to have a holographic model $\longrightarrow$ gravity with negative cosmological constant (AdS solution).

We want finite density $\longrightarrow$ $U(1)$ gauge field in the bulk.

One needs a notion of temperature. From AdS/CFT correspondence heating up the dual theory corresponds to having black hole in the bulk.

Finite temperature CFT $\longrightarrow$ Embedding the Schwarzschild black hole solution into AdS.

Black hole has temperature T which is identified with the temperature of dual CFT.

In general thermal geometry $\longrightarrow$ thermal field theory

Schwarzschild black hole solution into AdS$_5$

$$ds^2 = g dt^2 + \frac{dr^2}{g} + r^2 d\Omega_3^2, \qquad g = 1 + \frac{r^2}{R^2} - \frac{\mu}{r^2}$$

Here $r > r_+$, where r_+ are the largest root of $g = 0$.

The temperature is given by the period of the Euclidean time, $t \sim t + \beta$,

$$\beta = \frac{2\pi R^2 r_+}{2r_+^2 + R^2}$$

There is another solution which is $\mu = 0$. The temperature could be any value.

In the first solution the time is shrinking though it is not the case in the second one.

The are topologically different.

These two solutions make separate saddle point contributions to the thermal partition function (free energy=Euclidean action).

Which one is favored in the parameter space of the solutions?

$$I_1 - I_2 = \frac{\pi^3 r_+^2 (R^2 - r_+^2)}{4(2r_+^2 + R^2)}$$

It can change sign $\longrightarrow$ phase transit

In the dual gauge theory it corresponds to confinement/deconfinement phase transition

Which one we get $\longrightarrow$ what kind of holographic superconductor we are describing.

The last ingredient we need to construct our holographic model is a condensate.

In the bulk gravity it may be described by a scalar field coupled to gravity.

When the condensate is non-zero we should have a black hole with scalar hair!

To describe superconductor one needs a black hole solution which has hair at low temperature, though has no hair at high temperature.

It might seem puzzling due to no-hair theorems! but asymptotically AdS solution does the job.

Gubser showed that a simple model consisting of gravity, gauge field and charged scalar can exhibit such a behavior.

The simplest model is

$$S = \int d^4x\sqrt{g}\left[R + \frac{6}{L^2} - \frac{1}{4}F^2 - |\nabla\Psi - iqA\Psi|^2 - m^2\Psi^2\right]$$

where q is charge, and m is mass of the scalar.

This is a model which can describe a superconductor in 2+1 dimensions in hight temperature T_c.

The aim is not to derive the gravitational model from string theory. The idea is to find a gravity model with the properties we want.

The equations of motion

Scalar equation:

$$-(\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu)\Psi + m^2\Psi = 0$$

Maxwell's equations

$$\nabla^\mu F_{\mu\nu} = iq\Big[\Psi^*(\nabla_\nu - iqA_\nu)\Psi - \Psi(\nabla + iqA_\nu)\Psi^*\Big]$$

Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{3}{L^2}g_{\mu\nu} = T_{\mu\nu}$$

It is easy to see why black holes in this theory might be unstable to forming scalar hair: For an electrically charged black hole, the effective mass of Ψ is

$$m^2_{\text{eff}} = m^2 2 + q^2 g^{tt} A_t^2$$

This might change a sign!

We start with the following ansatz

$$ds^2 = -ge^{-\chi}dt^2 + \frac{dr^2}{g} + r^2(dx^2 + dy^2), \qquad A_t = \phi(r), \qquad \Psi = \psi(r)$$

$$\psi'' + (\frac{g'}{g} - \frac{\chi'}{2} + \frac{2}{r})\psi' - (\frac{m^2}{g} - \frac{q^2\phi^2 e^{\chi}}{g^2})\psi = 0,$$

$$\phi'' + (\frac{\chi'}{2} + \frac{2}{r})\phi' - \frac{2q^2\psi^2}{g}\phi = 0,$$

$$\chi' + r\psi'^2 + \frac{rq^2\phi^2\psi^2 e^{\chi}}{g^2} = 0,$$

$$\frac{1}{2}\psi' + \frac{\phi'^2 e^{\chi}}{4g} + \frac{g'}{gr} + \frac{1}{r^2} - \frac{3}{L^2 g} + \frac{m^2\psi^2}{2g} + \frac{q^2\psi^2\phi^2 e^{\chi}}{2g^2} = 0$$

We would like to work in probe limit where we can neglect the backreactions of the scalar field and gauge field on the background geometry.

Consider the limit

$$q \to \infty, \qquad qA,\ q\Psi \text{ finite}$$

In this limit χ=constant and the geometry is

$$ds^2 = -ge^{-\chi}dt^2 + \frac{dr^2}{g} + r^2(dx^2 + dy^2), \quad g = \frac{r^2}{L^2}(1 - \frac{r_0^3}{r^3})$$

In this limit the equations of gauge field and scalar field read

$$\psi'' + (\frac{g'}{g} + \frac{2}{r})\psi' - (\frac{m^2}{g} - \frac{\phi^2}{g^2})\psi = 0,$$

$$\phi'' + \frac{2}{r}\phi' - \frac{2\psi^2}{g}\phi = 0,$$

The aim is to solve these equations.

Boundary conditions at horizon:

$$\phi(r_0) = 0, \quad \text{to have finite energy at horizon}$$

$$g'(r_0)\psi'(r_0) = m^2\psi(r_0), \quad \text{from equation of notion}$$

As a result, even though we start with two second order equations which have a four parameter family of solutions, there is only a two parameter subfamily which is regular at the horizon.

They could be $\psi(r_0),\ \phi'(r_0)$.

Boundary conditions at infinity:

$$\psi = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + \cdots, \qquad \phi = \mu + \frac{\rho}{r} + \cdots$$

for $m^2 = -\frac{2}{L^2}$. One can consider solutions with $\psi^{(1)} = 0$ or $\psi^{(2)} = 0$.

we have one parameter family solutions.

What is the dual field theory?

The dual theory is a 2+1 dimensional conformal field theory (CFT) at temperature T.

The local gauge symmetry in the bulk corresponds to a global $U(1)$ symmetry in the CFT.

μ is the chemical potential and ρ is the charge density.

nonzero $\psi^{(i)}$ corresponds to a nonzero expectation value

$$\langle \mathcal{O}_i \rangle = \psi^{(i)}$$

Since we want the condensate to turn on without being sourced, we have set

$$\psi^{(i)} = \epsilon^{ij} \psi^{(j)}$$

We want to know how the condensate $\mathcal{O}_2$ behaves as a function of temperature.

One has to solve the equations, of course numerically. To do so we set $L = 1,\ r_0 = 1$.

We find that for $T < T_c$ the condensate is non-zero.

A nonzero condensate means that the black hole in the bulk has developed scalar hair.

One can compute the free energy (euclidean action) of these hairy configurations and compare with the solution $\psi = 0, \phi = \rho(1/r_0 - 1/r)$ which describes a black hole with the same charge or chemical potential, but no scalar hair.

It turns out that the free energy is always lower for the hairy configurations and becomes equal as $T \to T_c$.

The difference of free energies scales like $(T_c - T)^2$ near the transition, showing that this is a second order phase transition.

Conductivity

We want to compute the conductivity as a function of frequency.

According to the gauge/gravity dictionary, this is obtained by solving for fluctuations in the Maxwell field in the bulk.

Maxwell's equation for $A_x = A(r)e^{-i\omega t}$ gives (probe limit)

$$A'' + \frac{g'}{g}A' + (\frac{\omega^2}{g^2} - \frac{2\psi^2}{g})A = 0$$

We want to solve this with ingoing wave boundary conditions at the horizon, (retarded Green's function).

Asymptotically

$$A = A^{(0)} + \frac{A^{(1)}}{r} + \cdots$$

The limit of the electric field in the bulk is the electric field on the boundary: $E_x = -\dot{A}^{(0)}$.

the expectation value of the induced current is the first subleading term: $J_x = A^{(1)}$.

From Ohm's law we get:

$$\sigma(\omega) = \frac{J_x}{E_x} = -\frac{iA^{(1)}}{\omega A^{(0)}}$$

There is a pole at $\omega = 0$ in the $Im(\sigma)$, showing that there is a delta function in $Re(\sigma)$!

Some issues

• One may also consider the backreactions and solve the whole equations together.

Qualitatively we get the same results.

• Another issue one may address is the zero temperature limit

The extremal Reissner-Nordstrom AdS black hole has large entropy at $T = 0$.

If this was dual to a condensed matter system, it would mean the ground state was highly degenerate.

The extremal limit of the hairy black holes dual to the superconductor is not like Reissner-Nordstrom. It has zero horizon area, consistent with a nondegenerate ground state.

• Magnetic field

One of the characteristic properties of superconductors is that they expel magnetic fields.

A sufficiently strong field will destroy the superconductivity $\longrightarrow$ There is a critical field B_c.

Superconductors are divided into two classes depending on how they make the transition from a superconducting to a normal state as the magnetic field is increased.

• Type I superconductors: there is a first order phase transition at $B = B_c$, above which magnetic field lines penetrate uniformly and the material no longer superconducts. no longer superconducts.

• Type II superconductors: the magnetic field starts to penetrate the superconductor in the form of vortices with quantized flux when $B = B_{c1} < B_c$. The vortices become more dense as the magnetic field is increased, and at an upper critical field strength, $B = B_{c2} > B_c$, the material no longer superconducts.

- Vortices

At the onset of superconductivity when $B = B_{c2}$, there is a lattice of vortices.

Let us write the background metric using polar coordinates for the flat transverse space

$$ds^2 = -g dt^2 + \frac{dr^2}{g} + r^2(d\zeta^2 + \zeta^2 d\theta^2)$$

an ansatz

$$\Psi = \psi(r,\zeta)e^{in\theta}, \qquad A_t = A_t(r,\zeta), \qquad A_\theta = A_\theta(r,\zeta)$$

Substituting this into the field equations, one obtains a set of nonlinear PDE's.

The condensate is now a function of radius ζ; It vanishes at $\zeta = 0$, and approaches a constant at large ζ.

This shows that there is no superconductivity inside the vortex.

Other topics

p-wave superconductor; the condensate is a vector, i.e. $SU(2)$

Insulator/conductor/superconductor transitions

How to describe the pairing ?