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Ryu-Takayanagi Formula from Quantum Error Correction

A Review

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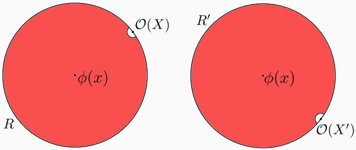
Introduction

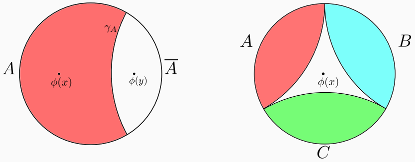
AdS/CFT & QEC

- AdS/CFT correspondence has recently been reinterpreted in the language of quantum error correcting codes. [Almehiri-Dong-Harlow '14/Pastawski-Yoshida-Harlow-Preskill '15/Hayden-Nezami-Qi-Walter-Yang '16/Harlow '16]
- This language naturally implements several features of the correspondence which were previously somewhat mysterious from the CFT point of view:
  1. **Radial Commutativity**:



[Harlow '18]

  2. **Subregion Duality**:



[Harlow '16]

3. **Ryu-Takayanagi Formula**: Given a CFT state ρ, we can define a boundary state ρ_A on any boundary subregion A. If ρ is "appropriate" then the von Neumann entropy of ρ_A is given by

$$\boxed{S(\rho_A) = \underbrace{\mathrm{Tr}\left(\rho \mathcal{L}_A\right)}_{\text{Area term}} + \underbrace{S_{bulk}(\rho_{\mathcal{E}_A})}_{\text{Bulk entropy}}.} \tag{1}$$

At leading order in Newton's constant G we have $\mathcal{L}_A = \frac{\mathrm{Area}(\gamma_A)}{4G}$, while at higher orders, both in G but also in other couplings such as α', there are corrections to $\mathcal{L}_A$ involving various quantities integrated on γ_A.

[Faulkner-Lewkowycz-Maldacena '13/Dong '14/Miao-Guo '15]

- It has been suggested that the RT formula might actually imply subregion duality in the entanglement wedge. [Almehiri-Dong-Harlow '14/Jaffris-Suh '14/Jaffris-Lewkowycz-Maldacena-Suh '15]
- For all three properties, a key point is that they hold only on a code subspace of states, which roughly speaking must be chosen to ensure that bulk effective field theory is a good approximation for the observables of interest throughout the subspace.

A Simple Example

Three-qutrit Code

- Let us begin with a simple example that illustrates many of the ideas of this paper: the three-qutrit code!
- The basic idea of quantum error correction is to protect a quantum state by encoding it into a code subspace of a larger Hilbert space. The three-qutrit code is an encoding of a single "logical" qutrit into the Hilbert space of three "physical" qutrits, with the code subspace $\mathcal{H}_{code}$ carrying the logical qutrit spanned by the basis

$$
\begin{aligned}
|\widetilde{0}\rangle &\equiv \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right)\\
|\widetilde{1}\rangle &\equiv \frac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right)\\
|\widetilde{2}\rangle &\equiv \frac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right).
\end{aligned}
$$

Three-qutrit Code

- This subspace has the property that there exists a unitary U_{12}, supported only on the first two qutrits, which obeys

$$U_{12}^{\dagger}|\widetilde{i}\rangle = |i\rangle_1 |\chi\rangle_{23}, \tag{2}$$

with

$$|\chi\rangle \equiv \frac{1}{\sqrt{3}} \left(|00\rangle + |11\rangle + |22\rangle\right). \tag{3}$$

Explicitly, U_{12} is a permutation that acts as

$$\begin{array}{lll} |00\rangle \to |00\rangle & |11\rangle \to |01\rangle & |22\rangle \to |02\rangle \\ |01\rangle \to |12\rangle & |12\rangle \to |10\rangle & |20\rangle \to |11\rangle \\ |02\rangle \to |21\rangle & |10\rangle \to |22\rangle & |21\rangle \to |20\rangle \end{array} . \tag{4}$$

- This protocol has two remarkable properties:
  1. For any state $|\widetilde{\psi}\rangle$, the reduced density matrix on any one of the qutrits is maximally mixed. Thus no single qutrit can be used to acquire any information about the state.
  2. The symmetry between the qutrits in the definition of $\mathcal{H}_{code}$ ensures that unitaries U_{13} and U_{23} will also exist, which means that the state $|\psi\rangle$ can be recovered on any two of the qutrits.

Three-qutrit Code

We can also phrase this correctability of single-qutrit erasures in terms of operators:

- Say that O is a linear operator on the single-qutrit Hilbert space. We can easily find a three-qutrit operator $\widetilde{O}$ that acts within $\mathcal{H}_{code}$ with the same matrix elements as O, but if we extend this operator arbitrarily on the orthogonal complement $\mathcal{H}_{code}^{\perp}$, then it will in general define an operator with support on all three physical qutrits.
- Using U_{12} however, we can define an operator

$$O_{12} \equiv U_{12} O_1 U_{12}^{\dagger} \tag{5}$$

  that acts within $\mathcal{H}_{code}$ in the same way as $\widetilde{O}$ but has support only on the first two qutrits.
- Again, by symmetry we can also define an O_{13} and O_{23}, so any logical operator on the code subspace can be represented as an operator with trivial support on any one of the physical qutrits.

- Now, say that we have an arbitrary mixed state $\widetilde{\rho}$ on $\mathcal{H}_{code}$, which is the encoding of a "logical" mixed state ρ:

$$\widetilde{\rho} = U_{12} \left(\rho_1 \otimes |\chi\rangle\langle\chi|_{23} \right) U_{12}^\dagger, \tag{6}$$

- So, defining

$$\begin{aligned} \widetilde{\rho}_3 &\equiv \mathrm{Tr}_{12}\, \widetilde{\rho} = \frac{1}{3} I_3, \\ \widetilde{\rho}_{12} &\equiv \mathrm{Tr}_3\, \widetilde{\rho} = U_{12} \left[\rho_1 \otimes \left(\frac{1}{3} I_3 \right) \right] U_{12}^\dagger, \end{aligned} \tag{7}$$

  we have the von Neumann entropies:

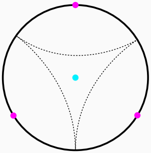
$$\begin{aligned} S(\widetilde{\rho}_3) &= \log 3 \\ S(\widetilde{\rho}_{12}) &= \log 3 + S(\widetilde{\rho}). \end{aligned} \tag{8}$$

- Once again, the symmetry ensures that analogous results hold for the entropies on other subsets of the qutrits.

Three-qutrit Code

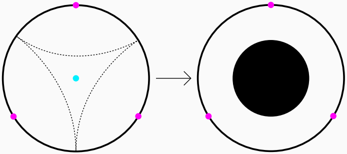
We can interpret this code as a model of AdS/CFT:

- The three physical qutrits are analogous to the local CFT degrees of freedom, and the code subspace $\mathcal{H}_{code}$ is analogous to the subspace where only effective field theory degrees of freedom are excited in the bulk.



[Harlow '18]

- The orthogonal complement $\mathcal{H}_{code}^{\perp}$ corresponds to the microstates of a black hole which has swallowed our point.



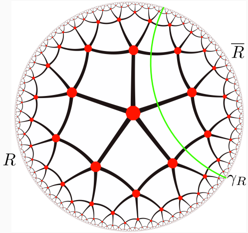
[Harlow '18]

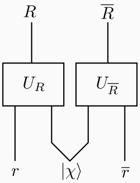
Three-qutrit Code

Now, let us see how this realizes the properties of AdS/CFT discussed before:

1. **Radial Commutativity:** We'd like to show that any "bulk local operator", meaning any operator $\widetilde{O}$ that acts within $\mathcal{H}_{code}$, commutes with all "local operators at the boundary", meaning it commutes with any operator that acts on only one physical qutrit. But O_{12}, O_{13}, and O_{23} each manifestly commute with boundary local operators on the third, second, or first qutrits respectively, and since they all act identically to $\widetilde{O}$ within the code subspace, it must be that within the code subspace $\widetilde{O}$ commutes with all boundary local operators. More precisely, if X is an operator on a single physical qutrit, and $|\widetilde{\psi}\rangle$,$|\widetilde{\phi}\rangle \in \mathcal{H}_{code}$, then $\langle\widetilde{\psi}|[\widetilde{O}, X]|\widetilde{\phi}\rangle = 0$.
2. **Subregion Duality:** We should think of x as being in the entanglement wedge of any two of the boundary qutrits. And indeed we see that any operator $\widetilde{O}$ can be represented on any two of the qutrits using O_{12}, O_{13}, or O_{23}.

3. **Ryu-Takayanagi Formula:** We have already computed the entropies (8). If we define an "area operator" $\mathcal{L}_{12} = \mathcal{L}_3 \equiv \log 3$, then apparently the RT formula (1) holds for any state $\widetilde{\rho}$ on the code subspace. This "area term" reflects the nontrivial entanglement in the state $|\chi\rangle$, while the "bulk entropy term" takes into account the possibility of the encoded qutrit being in a mixed state. The area term is essential for the functioning of the code, since if $|\chi\rangle$ were a product state, from (2) we see that the third qutrit would be extemporaneous, and there would be no way for both U_{23} and U_{13} to exist (one of them could exist if the first or second qutrit could access the state by itself).





[Harlow '18]

Algebraic encoding & RT

Useful advice:

SIAM Review
Vol. 9, No. 1, January, 1967

RANDOM MATRICES IN PHYSICS*

EUGENE P. WIGNER†

Introduction. It has been observed repeatedly that von Neumann made important contributions to almost all parts of mathematics with the exception of number theory. He had a particular interest in those parts of mathematics which formed cornerstones of other, more empirical sciences, such as physics or eco-

von Neumann In a Letter to Brikhoff (Nov. 6, 1935):

I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all Hilbert-space (as far as quantum-mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of

Von Neumann's conclusion:

This makes me strongly inclined, therefore, to take the ring of all bounded operators of Hilbert space ("Case I_∞" in our notation) less seriously, and Case II_1 more seriously, when thinking of an

Operator algebra QEC

We need to go from treating the bulk degrees of freedom in the entanglement wedge as a tensor factor $\mathcal{H}_r$ of $\mathcal{H}_{code}$ to viewing them as a subalgebra of the operators on $\mathcal{H}_{code}$:

- **Definition**

  Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ be the set of linear operators on $\mathcal{H}$. A subset $M \subset \mathcal{L}(\mathcal{H})$ is a von Neumann Algebra on $\mathcal{H}$ if the following hold:

  - *For all x, y in M, we have $xy \in M$*
  - *For all x, y in M, we have $x + y \in M$*
  - *For all x in M, we have $x^\dagger \in M$*
  - *For all $\lambda \in \mathbb{C}$, we have $\lambda I \in M$*

  In other words, a von Neumann algebra is is a subset of the operator algebra on $\mathcal{H}$ which is closed under addition, multiplication, and complex conjugation, and which contains all scalar multiples of the identity.

Operator algebra QEC

Any von Neumann algebra on $\mathcal{H}$ induces two other natural von Neumann algebras on $\mathcal{H}$:

- Let M be a von Neumann algebra on $\mathcal{H}$. Its commutant, M', is the von Neumann algebra

$$M' \equiv \{x \in \mathcal{L}(\mathcal{H}) | xy = yx \quad \forall y \in M\}. \tag{9}$$

- Let M be a von Neumann algebra on $\mathcal{H}$. Its center, Z_M, is the von Neumann algebra

$$Z_M \equiv M \cap M'. \tag{10}$$

- One simple example of these arises when the Hilbert space tensor factorizes as $\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}_{\bar{r}}$ and we take M to be all the operators on $\mathcal{H}_r$. We then have:

$$\begin{aligned} M &= \mathcal{L}(\mathcal{H}_r) \otimes I_{\bar{r}} \\ M' &= I_r \otimes \mathcal{L}(\mathcal{H}_{\bar{r}}) \\ Z_M &= \lambda I_{r\bar{r}}. \end{aligned} \tag{11}$$

  In this situation we say that M is a factor, and we see that any factor has a trivial center consisting of only scalar multiples of the identity.

Operator algebra QEC

- **Theorem**

  Let M be a von Neumann algebra on a finite-dimensional Hilbert space $\mathcal{H}$. Then there exists a Hilbert space direct sum decomposition

$$\mathcal{H} = \oplus_\alpha \Big(\mathcal{H}_{r_\alpha} \otimes \mathcal{H}_{\bar{r}_\alpha} \Big) \tag{12}$$

  such that we have

$$\begin{aligned} M &= \oplus_\alpha \Big(\mathcal{L}(\mathcal{H}_{r_\alpha}) \otimes I_{\bar{r}_\alpha} \Big) \\ M' &= \oplus_\alpha \Big(I_{r_\alpha} \otimes \mathcal{L}(\mathcal{H}_{\bar{r}_\alpha}) \Big) \\ Z_M &= \oplus_\alpha \Big(\lambda_\alpha I_{r_\alpha \bar{r}_\alpha} \Big). \end{aligned} \tag{13}$$

  - In other words, every von Neumann algebra on a finite-dimensional Hilbert space is a block diagonal direct sum of factors.

- von Neumann's definition of the entropy has a natural extension to an entropy of a state ρ on a general von Neumann algebra M. The basic idea is to consider the diagonal blocks of ρ in the block decomposition (12):

$$\rho = \begin{pmatrix} p_1 \rho_{r_1 \bar{r}_1} & \cdots & \cdots \\ \vdots & p_2 \rho_{r_2 \bar{r}_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{14}$$

  where we have extracted a positive coefficient p_α from each diagonal block so that $\mathrm{Tr}\, \rho_{r_\alpha \bar{r}_\alpha} = 1$. The p_α obey $\sum_\alpha p_\alpha = 1$.
- We then define the entropy of the state ρ on M as

$$\boxed{S(\rho, M) = -\sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha S(\rho_{r_\alpha}),} \tag{15}$$

  where $\rho_{r_\alpha} = \mathrm{Tr}_{\bar{r}_\alpha}\, \rho_{r_\alpha \bar{r}_\alpha}$.
- This entropy has two contributions: a "classical" piece associated to the uncertainty over which block we are in, and a "quantum" piece where we average the original von Neumann entropy over blocks.

Operator algebra QEC

- There is also an algebraic version of the relative entropy of two states ρ and σ on an algebra M, which when M is a factor is defined as

$$S(\rho|\sigma, M) = \mathrm{Tr}\left(\rho_r \log \rho_r\right) - \mathrm{Tr}\left(\rho_r \log \sigma_r\right) \equiv S(\rho_r|\sigma_r). \tag{16}$$

- Relative entropy is a measure of how much the states ρ and σ can be distinguished by measuring elements of M.
- Its definition for an arbitrary von Neumann algebra M is

$$\boxed{S(\rho|\sigma, M) = \sum_\alpha p_\alpha^{\{\rho\}} \log \frac{p_\alpha^{\{\rho\}}}{p_\alpha^{\{\sigma\}}} + \sum_\alpha p_\alpha^{\{\rho\}} S\left(\rho_{r_\alpha}|\sigma_{r_\alpha}\right).} \tag{17}$$

Theorem

Let $\mathcal{H}$ be a finite dimensional Hilbert space which tensor factorizes into $\mathcal{H}_R \otimes \mathcal{H}_{\overline{R}}$, let $\mathcal{H}_{code}$ be a subspace of $\mathcal{H}$, and let M be a von Neumann algebra on $\mathcal{H}_{code}$. Then the following three statements are equivalent:

(1) *For all operators $\widetilde{O} \in M$ and $\widetilde{O}' \in M'$, there are operators $O_R \in \mathcal{L}(\mathcal{H}_R)$ and $O'_{\overline{R}} \in \mathcal{L}(\mathcal{H}_{\overline{R}})$ such that for all states $|\widetilde{\psi}\rangle \in \mathcal{H}_{code}$ we have*

$$\begin{aligned} O_R|\widetilde{\psi}\rangle &= \widetilde{O}|\widetilde{\psi}\rangle \\ O_R^\dagger|\widetilde{\psi}\rangle &= \widetilde{O}^\dagger|\widetilde{\psi}\rangle \\ O'_{\overline{R}}|\widetilde{\psi}\rangle &= \widetilde{O}'|\widetilde{\psi}\rangle \\ O'^\dagger_{\overline{R}}|\widetilde{\psi}\rangle &= \widetilde{O}'^\dagger|\widetilde{\psi}\rangle \end{aligned} \tag{18}$$

(2) *There exists an operator $\mathcal{L}_R \in Z_M$ such that for any state $\widetilde{\rho}$ on $\mathcal{H}_{code}$, we have*

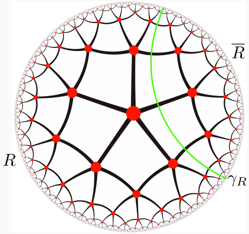
$$\begin{aligned} S(\widetilde{\rho}_R) &= \mathrm{Tr}\,(\widetilde{\rho}\mathcal{L}_R) + S(\widetilde{\rho}, M) \\ S(\widetilde{\rho}_{\overline{R}}) &= \mathrm{Tr}\,(\widetilde{\rho}\mathcal{L}_R) + S(\widetilde{\rho}, M'). \end{aligned} \tag{19}$$

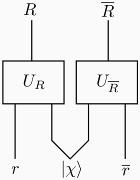
Subregion duality & RT formula

(3) For any states $\widetilde{\rho}$, $\widetilde{\sigma}$ on $\mathcal{H}_{code}$, we have

$$\begin{aligned} S(\widetilde{\rho}_R|\widetilde{\sigma}_R) &= S(\widetilde{\rho}|\widetilde{\sigma}, M) \\ S(\widetilde{\rho}_{\overline{R}}|\widetilde{\sigma}_{\overline{R}}) &= S(\widetilde{\rho}|\widetilde{\sigma}, M') \end{aligned} \tag{20}$$

- The application of this theorem to AdS/CFT works as follows: $\mathcal{H}$ is the full Hilbert space of the boundary CFT, and $\mathcal{H}_R$ and $\mathcal{H}_{\overline{R}}$ describe the CFT degrees of freedom in a boundary subregion R and its complement $\overline{R}$.





[Harlow '18]

Consequences for holography

Consequences for holography

This theorem has many important consequences for holography, and I do not have time to really do any of them justice. I will instead just sketch of a few:

- Since the Ryu-Takayanagi formula has been independently established using the replica trick methodology, the mentioned theorem establishes subregion duality in the entanglement wedge once and for all: we now know precisely which bulk subregion is dual to any boundary subregion. So far this is probably the biggest achievement of the quantum error correction perspective on holography.
- We see that to get a nontrivial area operator, meaning an area operator which is not proportional to the identity, it is essential that we take M to be an algebra with nontrivial center. In the bulk this is related to the fact that there are diffeomorphism constraints which present the Hilbert space from factorizing.

Consequences for holography

- We can use the framework of operator algebra quantum error correction with complementary recovery to study what happens if we include superpositions of geometries and/or black holes in the code subspace. In such situations it is important to discuss properly-dressed diffeomorphism-invariant observables, but these are naturally accommodated in an algebraic framework. In the former case we see that there is no problem with the RT formula continuing to hold in superpositions of geometries, which here roughly speaking correspond to states with projections onto more than one α.
- Part of the choice of code subspace in the CFT is dual to the choice of short-distance cutoff in the bulk effective field theory. For example in equation (19), the left hand side depends only on the state $\widetilde{\rho}$ and the region R, it does not depend on our choice of code subspace, while both terms on the right hand side do depend on the choice of code subspace. This is a coding analogue of the standard observation that entropy can be passed between the bulk and the area terms by changing the UV cutoff.

Questions?

Thank you!