Propagation of Entanglement in Scale Invariant Field Theories

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IPM

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This talk is based on:

In collaboration with Reza Mozaffar,

- "Entanglement in Lifshitz-type Quantum Field Theories"
  arXiv: 1705.00483 [hep-th],
  JHEP 1707 (2017) 120

- "Logarithmic Negativity in Lifshitz Harmonic Models"
  arXiv: 1712.03731 [hep-th],
  J.Stat.Mech. 1805 (2018) no.5, 053113

- "Entanglement Evolution in Lifshitz-type Scalar Theories"
  arXiv: 1811.11470 [hep-th],
  JHEP 1901 (2019) 137

Lifshitz Symmetry

- Lifshitz scaling [Lifshitz '41, Hertz '76]

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- Algebra: standard Poincare algebra between H, P_i and J_{ij} together with

$$[D, J_{ij}] = 0 \quad , \quad [D, P_i] = i\, P_i \quad , \quad [D, H] = i\, z\, H$$

where

$$H = -i\partial_t \quad , \quad J_{ij} = -i(x_i \partial_j - x_j p_i)$$
$$P_i = -i\partial_i \quad , \quad D = -i(z\, t \partial_t + x^i \partial_i)$$

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- Anisotropic scaling: $z\, T^0{}_0 + T^i{}_i = 0$

Lifshitz Scalar Theory

- Scalar Theory with Lifshitz symmetry $(m \to 0)$

$$I = \frac{1}{2} \int dt d\vec{x} \left[\dot{\phi}^2 - \sum_{i=1}^{d} (\partial_i^z \phi)^2 - m^{2z} \phi^2 \right],$$

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- Ground state 2-pt function $(m \to 0)$

$$\langle\phi(0)\phi(r)\rangle \sim r^{-d+z}.$$

Lifshitz Harmonic Lattice

- Lifshitz Hamiltonian density $(\pi = \partial\mathcal{L}/\partial\dot{\phi})$

$$\mathcal{H}_{\text{Lif.}} = \frac{1}{2}\left(\pi^2 + \sum_{i=1}^{d}(\partial_i^z\phi)^2 + m^{2z}\phi^2\right)$$

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- Lifshitz harmonic lattice [Mozaffar-AM '17; He-Magan-Vandoren '17]

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- Straightforwardly generalizes to d-dimension

Oscillator Chains

- Lorentzian case ($z = 1$)

$$H_{\mathrm{LHL}}(1) = \frac{1}{2}\sum_{n=0}^{N-1}\left[\pi_n^2 + (\phi_{n+1} - \phi_n)^2 + m^2\phi_n^2\right]$$

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- $z > 1$ case:

$$H_{\mathrm{LHL}}(2) = \frac{1}{2}\sum_{n=0}^{N-1}\left[\pi_n^2 + m^4\phi_n^2 + (\phi_{n+1} - 2\phi_n + \phi_{n-1})^2\right]$$

$$H_{\mathrm{LHL}}(3) = \frac{1}{2}\sum_{n=0}^{N-1}\left[\pi_n^2 + m^6\phi_n^2 + (\phi_{n+2} - 3\phi_{n+1} + 3\phi_n - \phi_{n-1})^2\right]$$

$$H_{\mathrm{LHL}}(4) = \frac{1}{2}\sum_{n=0}^{N-1}\left[\pi_n^2 + m^6\phi_n^2 + (\phi_{n+2} - 4\phi_{n+1} + 6\phi_n - 4\phi_{n-1} + \phi_{n-2})^2\right]$$

Lifshitz Harmonic Lattice

- In momentum space

$$H_{\text{Lif.}} = \sum_k \omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right) \quad , \quad \omega_k^2 = m^{2z} + \left(2 \sin \frac{\pi k}{N} \right)^{2z}$$

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- Correlators (with periodic B.C.)

$$\langle \phi_i \phi_j \rangle = \frac{1}{2N} \sum_{k=0}^{N} \omega_k^{-1} \left(2n(k) + 1 \right) \cos \left(\frac{2\pi (i-j) k}{N} \right),$$

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$$n(k) := \langle a_k^\dagger a_k \rangle \quad , \quad \forall \, |\psi\rangle \; : \; \langle a_k a_{k'} \rangle = \langle a_k^\dagger a_{k'}^\dagger \rangle = 0$$

Outline

1. Entanglement Entropy
   - Area Law?
   - Additivity Corner Contributions?
2. Relaxation After a Quench
   - Global Quantum Quenches
   - 2-d CFT
3. Quasi-particle Picture
   - Monochromatic Quasi-Particles
   - Tower of Quasi-Particles
4. Propagation of Ent. in Lifshitz Theories
   - Quasi-Particles in Lifshitz Theory
   - Widen Light-cones

Entanglement Entropy

Peschel's Correlator Method

- For restricted operators $\mathcal{O}_A$

$$\mathrm{Tr}\left[\rho_A \mathcal{O}_A\right] = \langle \mathcal{O}_A \rangle$$

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$$\rho_A = \mathcal{N} \exp\left(\sum_i \varepsilon_i a_i^\dagger a_i\right) \quad , \quad \varepsilon_k = 2\coth^{-1}(2\nu_k)$$

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- Entanglement measures [Casini,Huerta '09]

$$S_A = \sum_{k=1}^{n_A} \left[\left(\nu_k + \frac{1}{2}\right)\log\left(\nu_k + \frac{1}{2}\right) - \left(\nu_k - \frac{1}{2}\right)\log\left(\nu_k - \frac{1}{2}\right)\right],$$

$$S_A^{(n)} = \frac{1}{n-1}\sum_{k=1}^{n_A} \log\left[\left(\nu_k + \frac{1}{2}\right)^n - \left(\nu_k - \frac{1}{2}\right)^n\right],$$

(1+1)-dimensions (massless)

- In local field theories

  [Callan,Wilczek '94; Calabrese,Cardy '04]

$$S(\ell) = \mathcal{S}_{\text{univ.}} \log \frac{\ell}{\epsilon} + S_0$$

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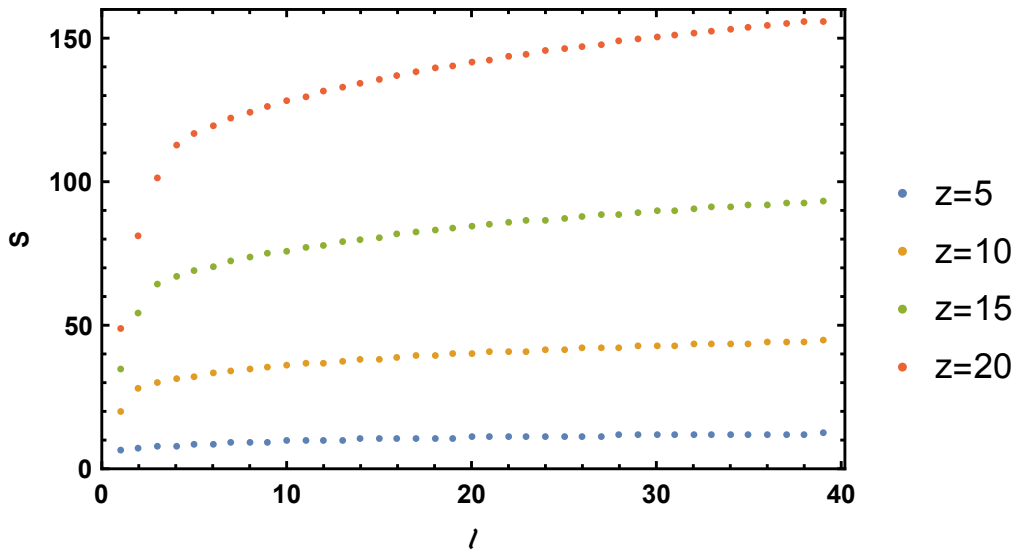
- Logarithmic behaviour



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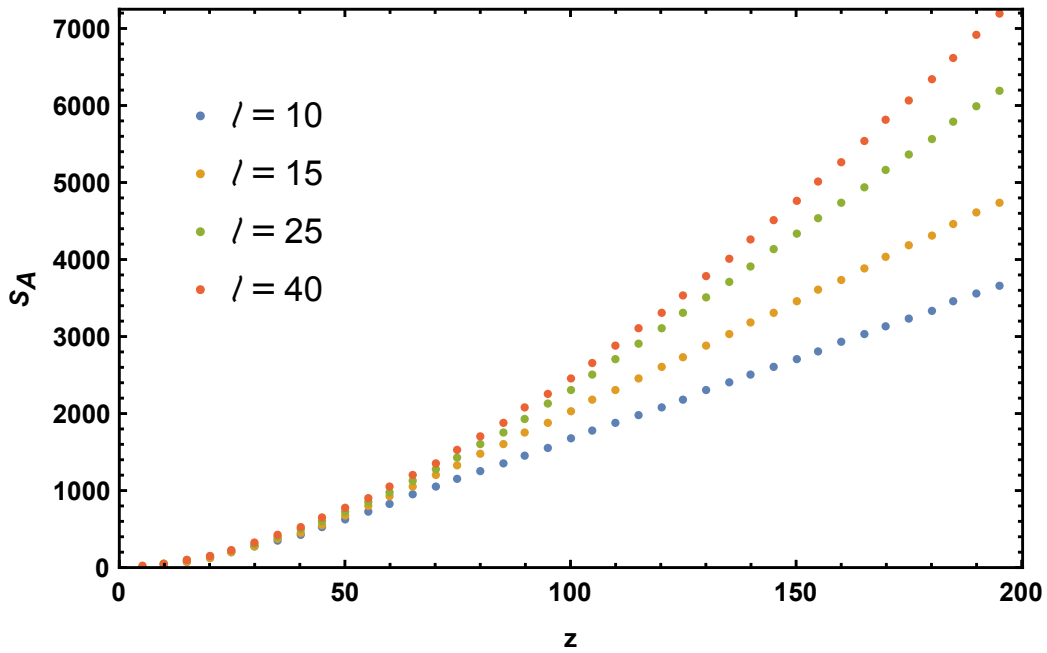
- How entropy depends on z?

$$S_{\text{leading}}(z) \propto \begin{cases} z^2 & z \lesssim \ell \\ z & z \gg \ell \end{cases}$$



Analytical Results: Periodic Subregions

- Periodic subregions: $N_A = N/p$ and $p \in \mathbb{Z}^+$

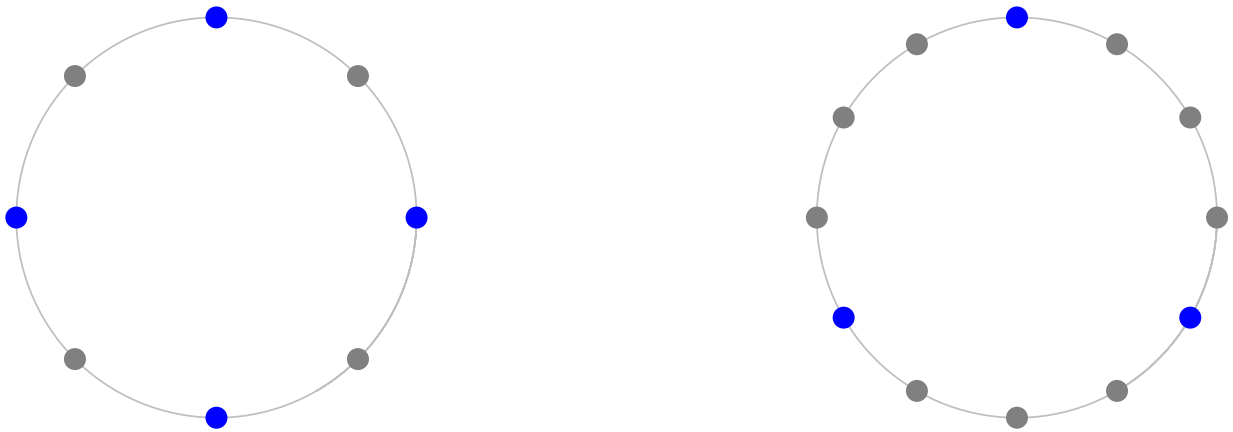


Figure: Left: $N = 8$, $p = 2$. Right: $N = 12$, $p = 4$.

[He-Magan-Vandoren '16]

Analytical Results: Periodic Subregions

- In continuum $(N, N_A \gg 1)$ $x = N/\ell$

$$\nu(x) = \frac{1}{2p} \left(\sum_{j,k=0}^{p-1} \frac{\omega\left(x + \frac{k-1/2}{p}\right)}{\omega\left(x + \frac{j}{p}\right)} \right)^{\frac{1}{2}} \quad , \quad \omega^2(x) = m^{2z} + (2 \sin \pi x)^{2z}$$

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- The entropy

$$\frac{S_A}{N_A} = p \int_0^{1/p} dx \left[\left(\nu(x) + \frac{1}{2}\right) \log\left(\nu(x) + \frac{1}{2}\right) - \left(\nu(x) - \frac{1}{2}\right) \log\left(\nu(x) - \frac{1}{2}\right) \right]$$

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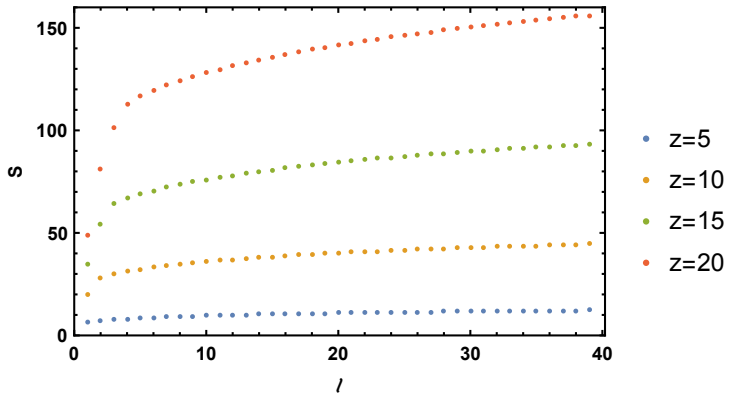
- For $p = 2$ in $z \to \infty$

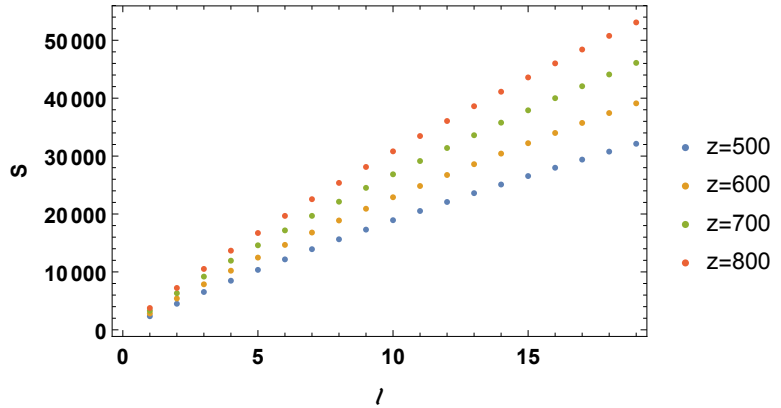
$$\frac{S_A}{N_A} = \frac{2G_C}{\pi} z + \mathcal{O}\left(z^0\right)$$

confirmation of linear regime

Area Law?

- Logarithmic versus linear scaling





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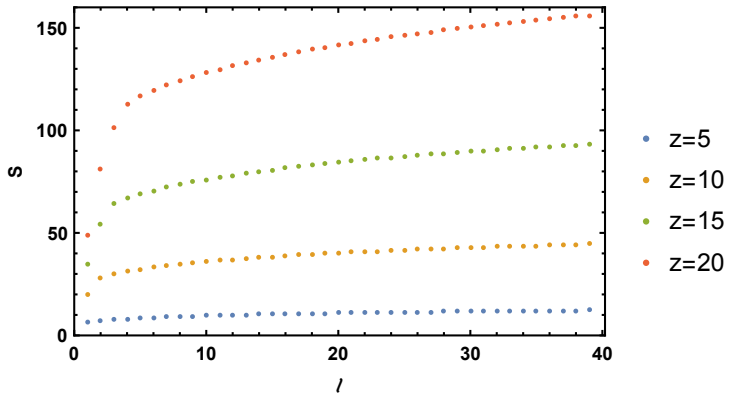

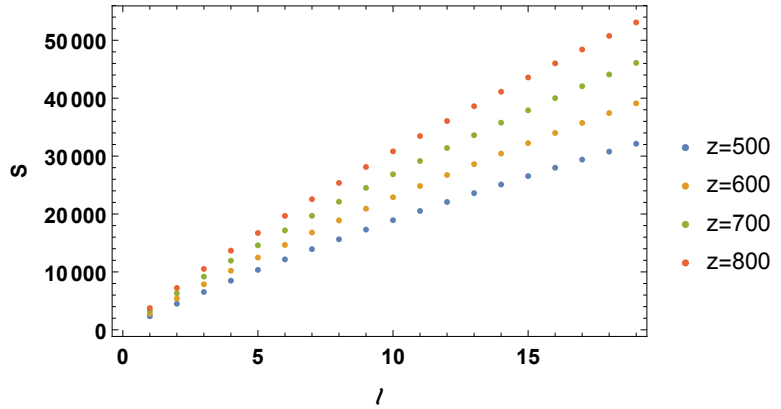

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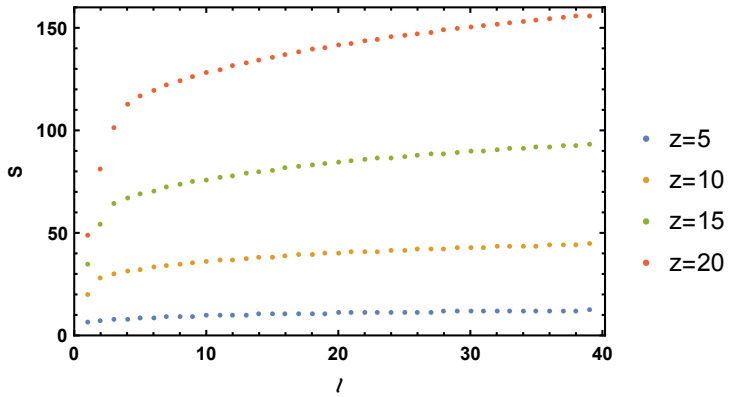

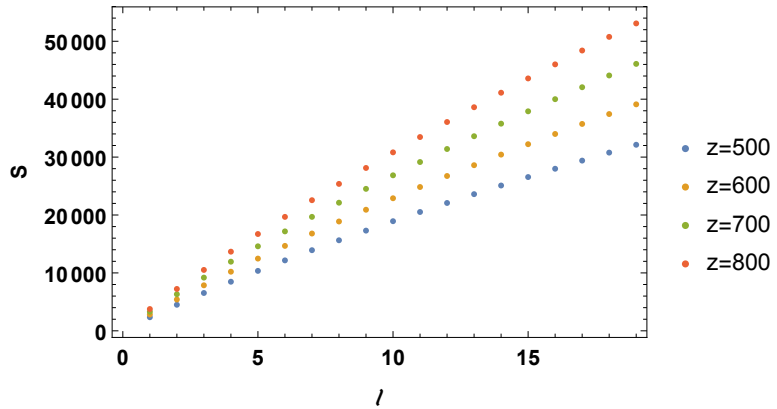

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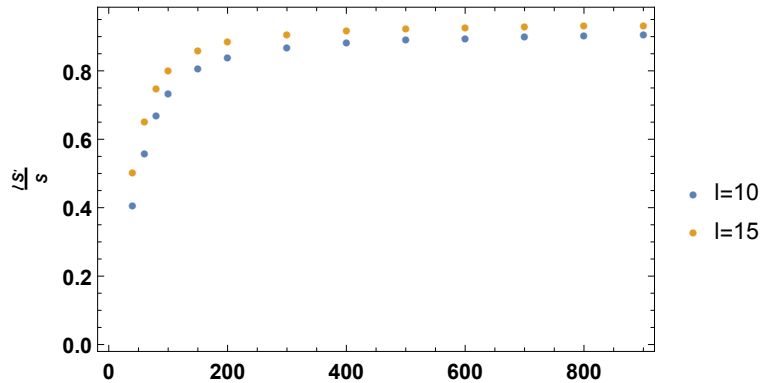
- Is it really volume law? Consider $S \sim \ell^{\alpha}$



What Happens to $S_A = S_B$?

- For any pure (including the vacuum) state one must have

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Strongest support for area law [Srednicki '93]

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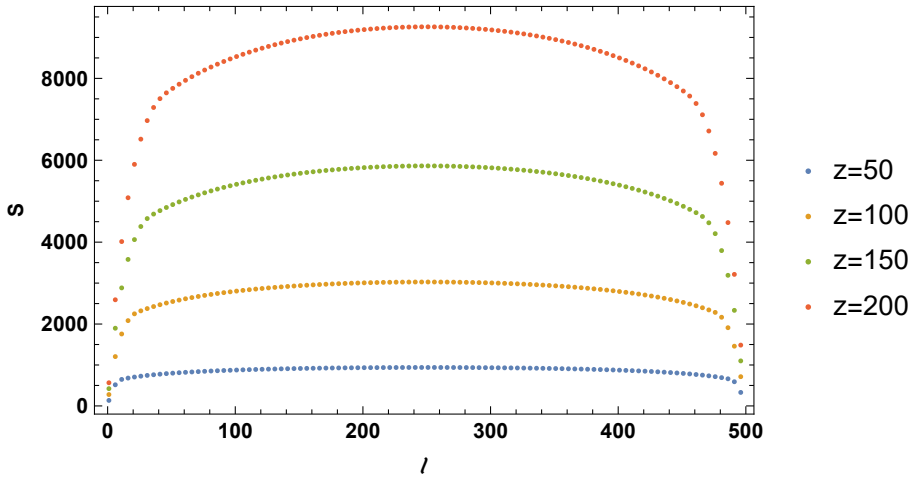

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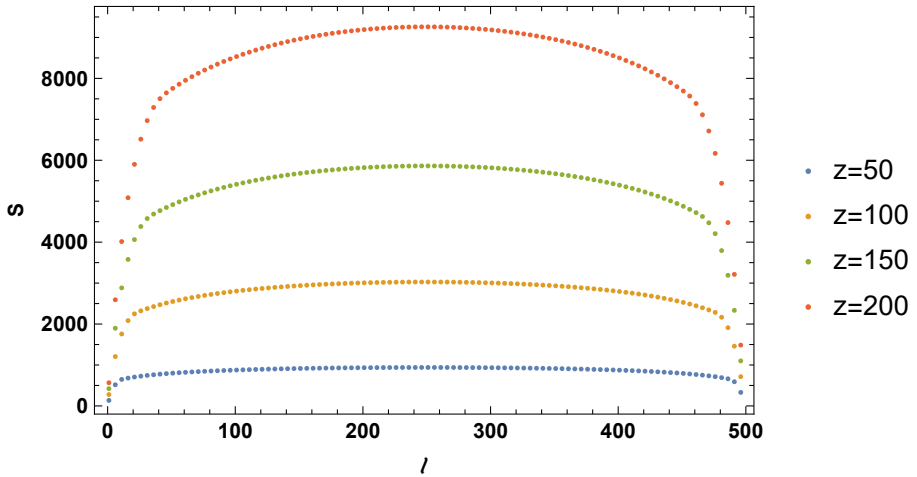

- For A's which S_A scales with V_A,

$$S_B \not\sim \text{Area}[B] \quad , \quad S_B \sim \text{Vol}[A]$$

(2+1)-dimensions (massless)

- In local field theories

$$S(\ell) = \mathcal{S}_{\text{area}} \frac{\ell}{\epsilon} + \mathcal{S}_{\text{corner}} \log \frac{\ell}{\epsilon} + S_0$$

[Casini-Huerta '06; Hirata-Takayanagi '06; Myers-Singh '12]

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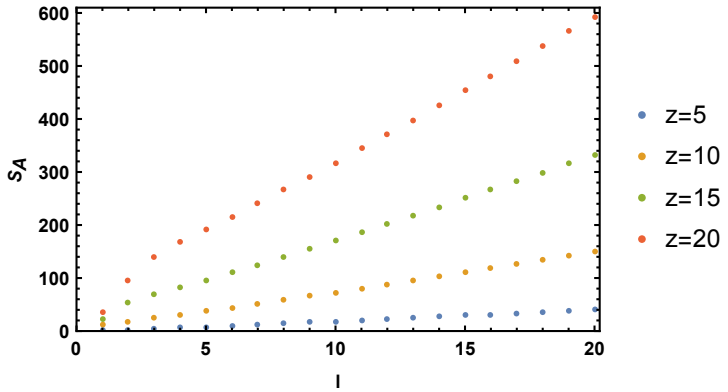
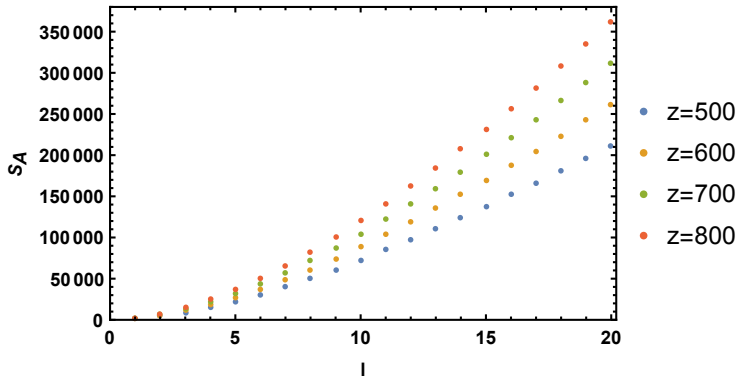
- linear versus quadratic scaling





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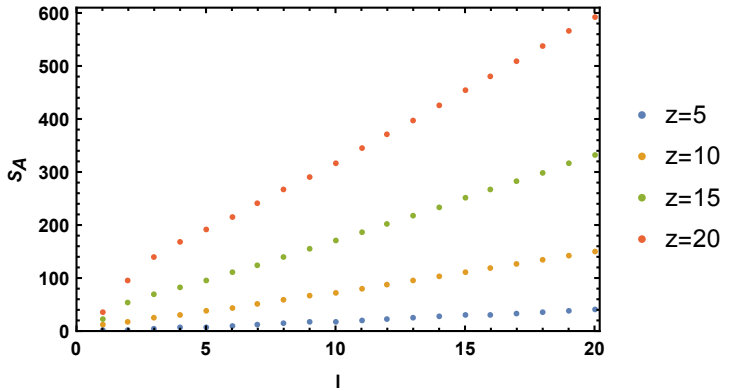

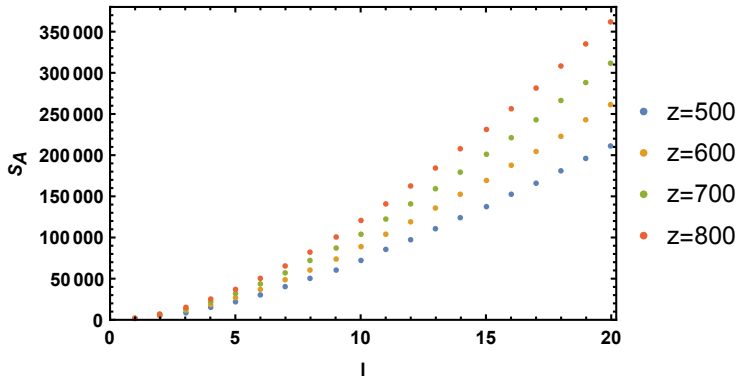

- Our fit function

$$S^{(z)}(\ell) = \mathcal{S}^{(z)}_{\text{vol.}} \frac{\ell^2}{\epsilon^2} + \mathcal{S}^{(z)}_{\text{area}} \frac{\ell}{\epsilon} + \mathcal{S}_{\text{corner}}(z) \log \frac{\ell}{\epsilon} + S^{(z)}_0$$

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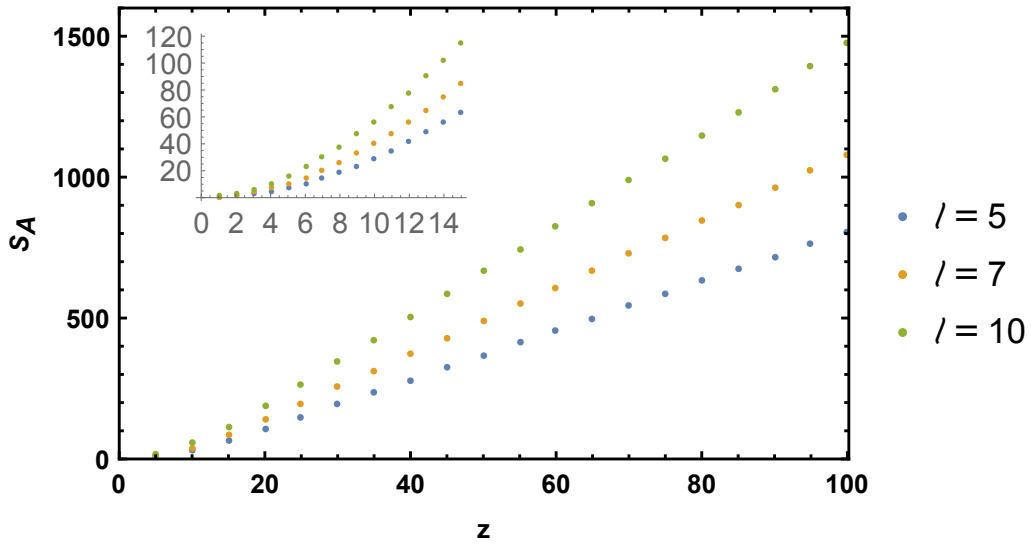

Shape Dependence of S_A

- In local theories: $\mathcal{S}_{\text{corner}} = \sum_{\text{corners on } \partial \text{A}} a\left(\theta_i\right)$

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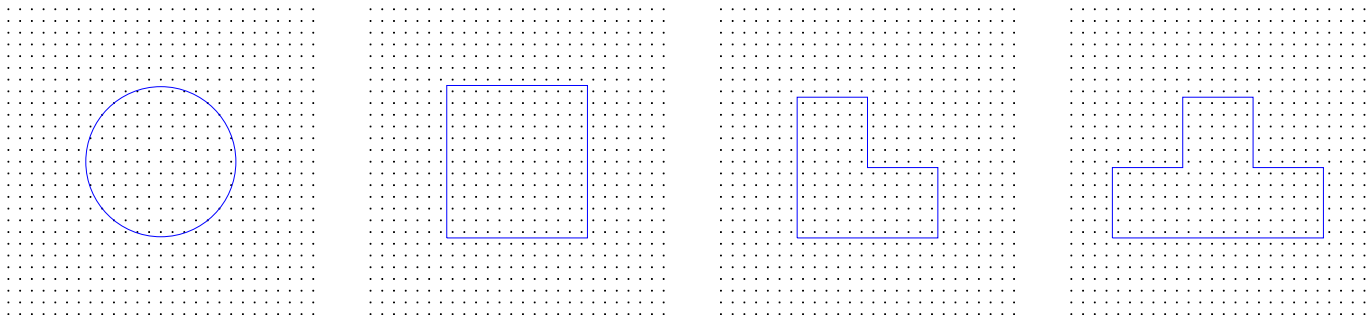
- Consider these geometries



Shape Dependence of S_A

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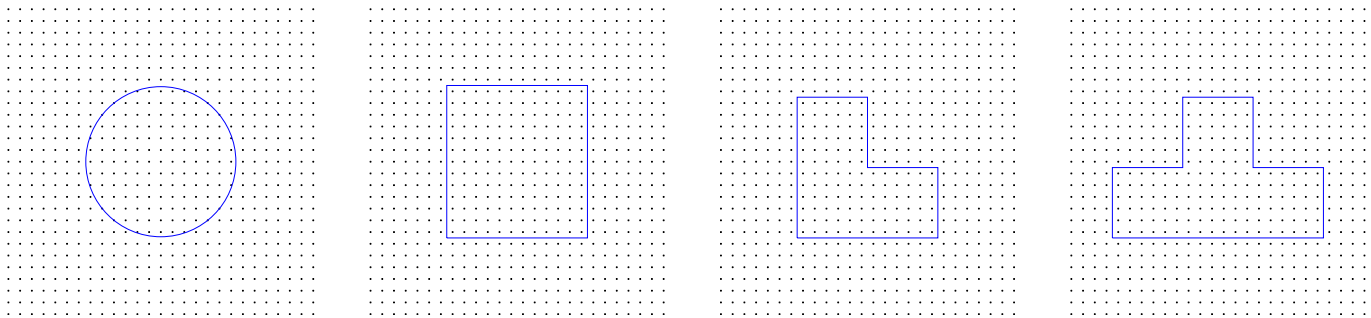

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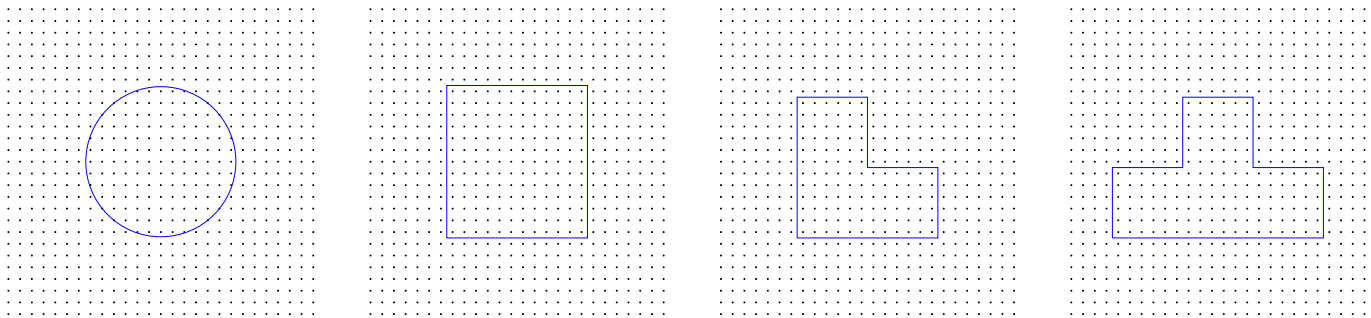

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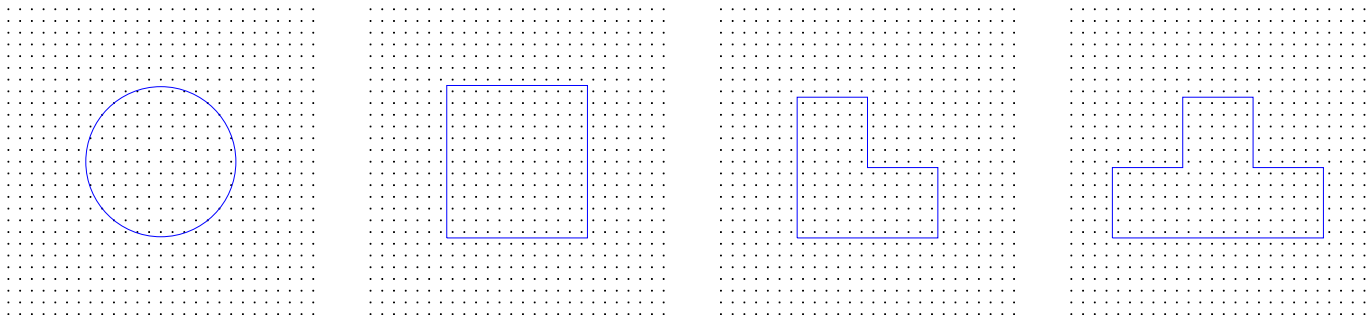

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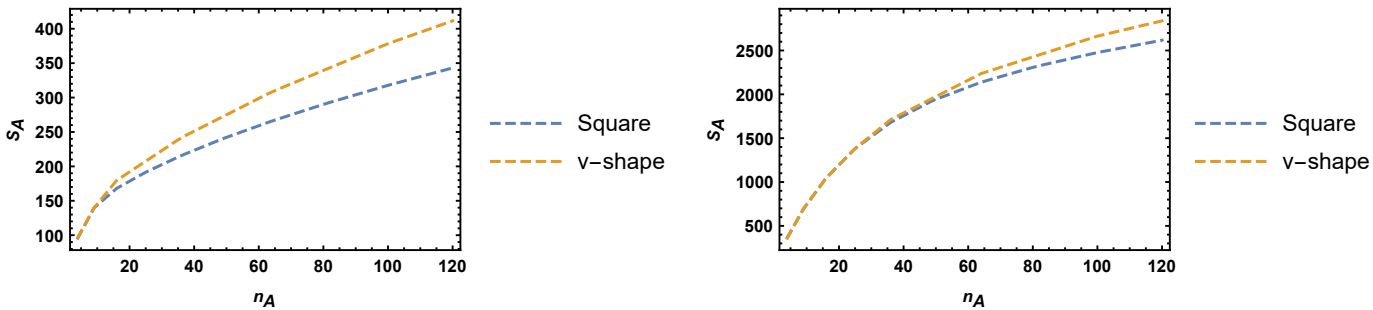

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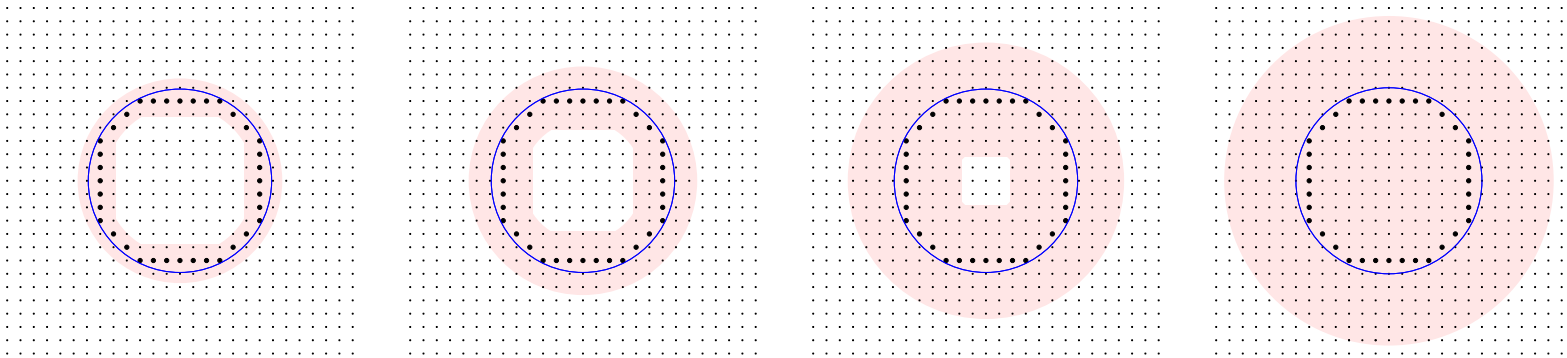

$z = 1$ $z = 2$ $z = 4$ $z = 6$

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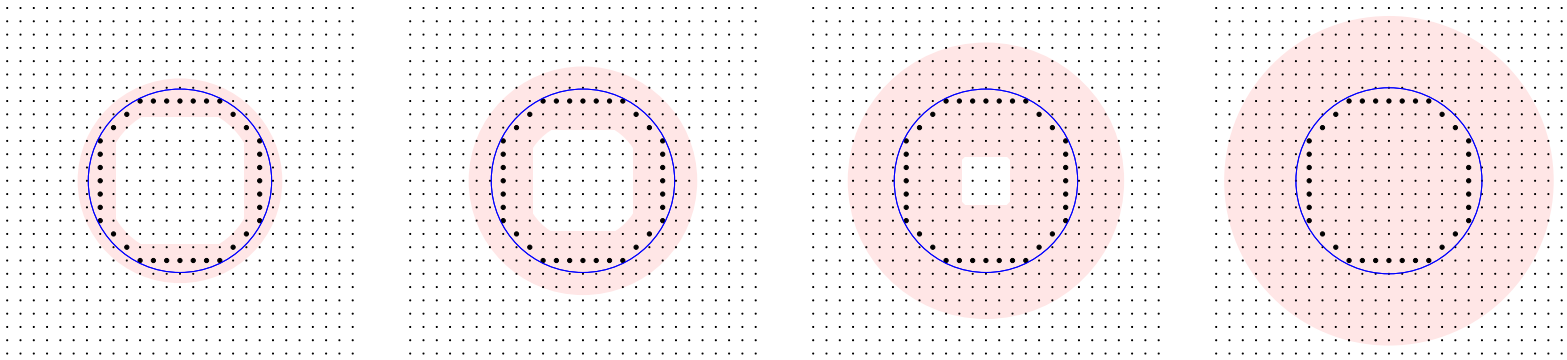
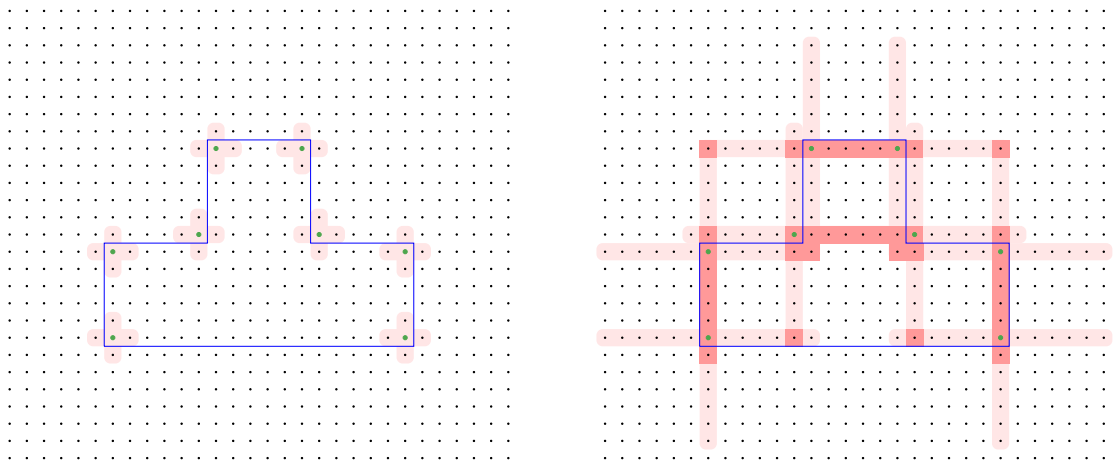


- What happens at the corners



Relaxation After a Quench

Global Quantum Quenches

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- Does the system reach local equilibrium?
- Central object to study local equilibrium is

$$\rho_A(t) = \mathrm{Tr}_{\bar{A}} |\psi(t)\rangle\langle\psi(t)|$$

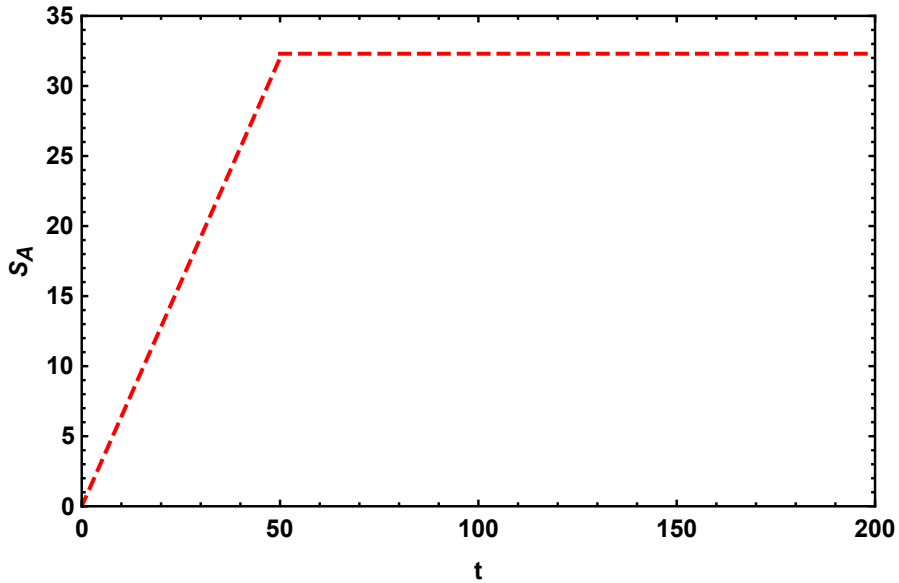
Simplest Case: 2-d CFT

- Consider the following Pre(Post) Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + (\partial\phi)^2 + m(t)^2\phi^2\right)$$

where $m(t) = m_0\Theta(-t)$

- Propagation of Entanglement in $\delta t \ll \ell_x$ limit



Quasi-particle Picture

Monochromatic Free Streaming Quasi-Particles

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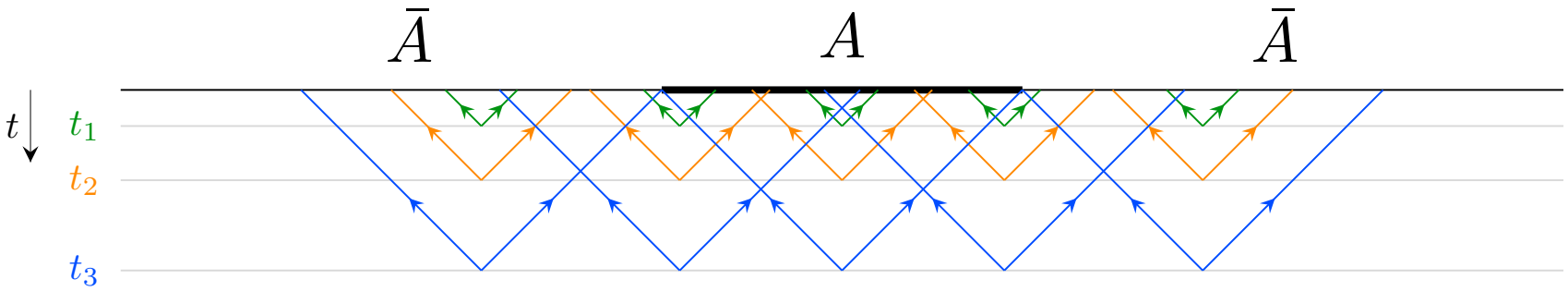
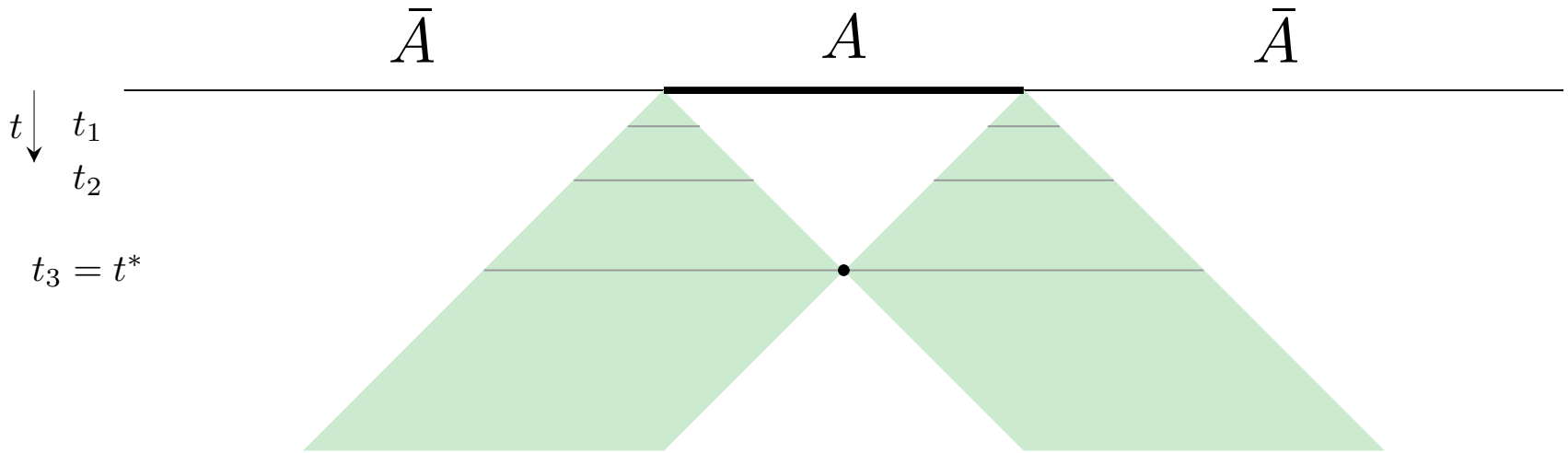


Monochromatic Free Streaming Quasi-Particles

- Contribution to entanglement entropy:



- Definition of saturation time, t^*, (for a single interval) is the intersect point of light-cones from the endpoints of the interval

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- In this specific case (on the lattice) we have

$$v_g(k) = \pm\frac{\sin k}{\sqrt{m^2 + \left(2\sin\frac{k}{2}\right)^2}} \quad , \quad v_g^{\max} = \pm\sqrt{1 + \frac{m^2}{2} - m\sqrt{1 + \left(\frac{m}{2}\right)^2}}$$

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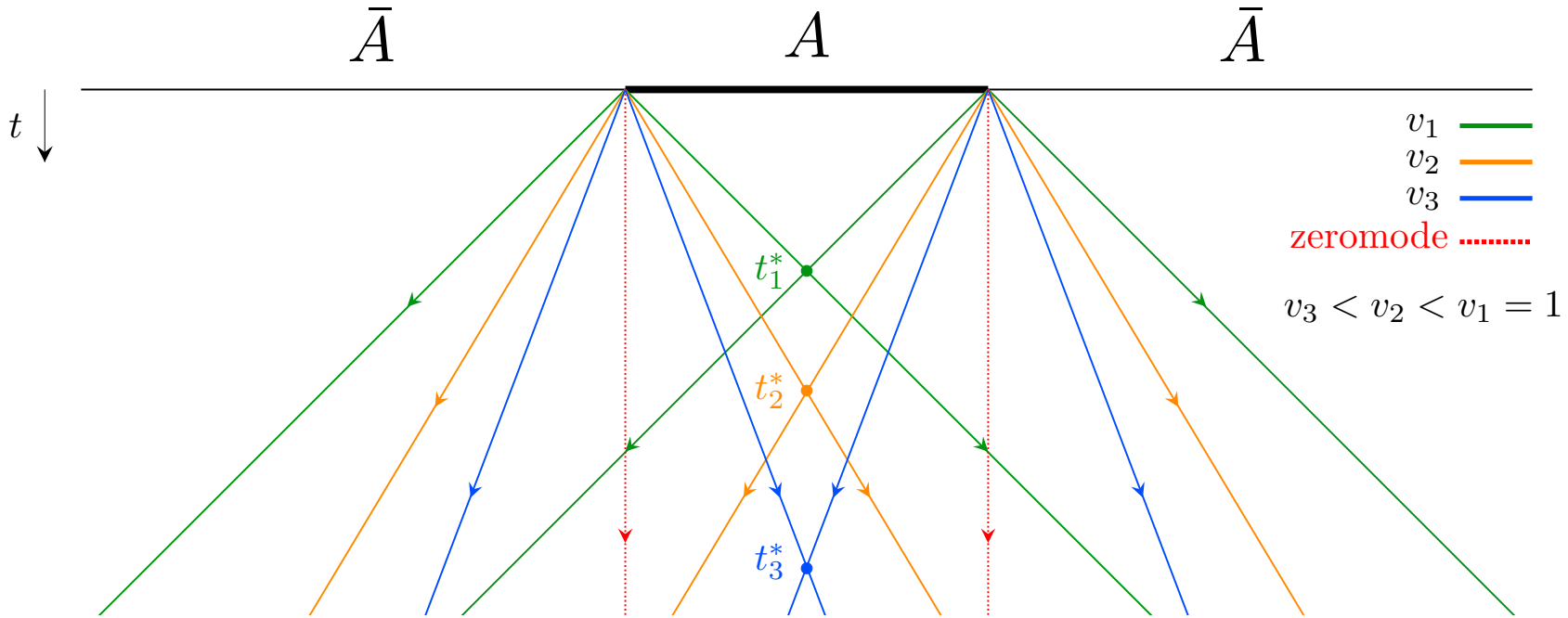
- Each mode saturates at $t^*(k) = \frac{\ell}{2v_g(k)}$

Free Streaming Quasi-Particles

1. Assume post-quench mass m_1 has a tiny value $\Rightarrow \quad v_g^{\max} \approx 1$



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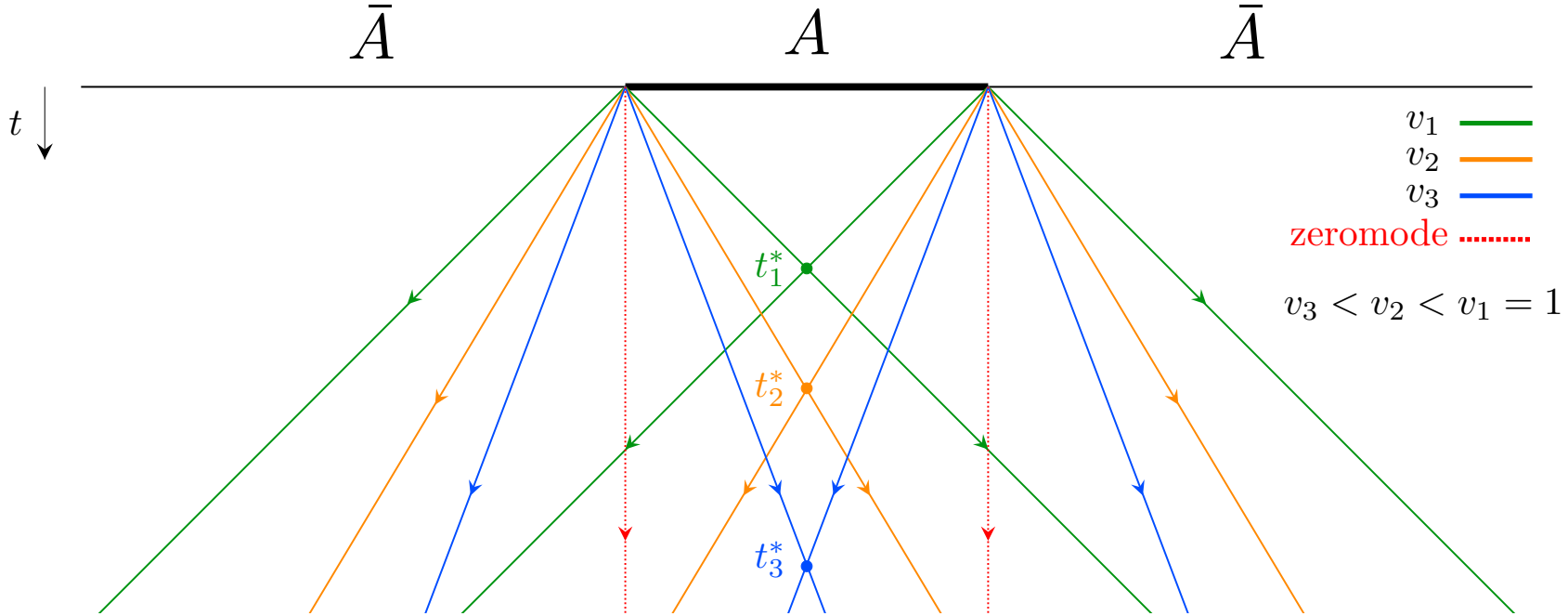

2. Assume different post-quench mass values $m_1 < m_2 < m_3$,
$\Rightarrow \quad 1 > v_g^{\max}(m_1) > v_g^{\max}(m_2) > v_g^{\max}(m_3)$

Alba-Calabrese Quasi-Particle Picture

- Alba-Calabrese [18'] : a spectrum of QPs with $v_g(k)$

$$\boxed{S(t) = 2t \int_{2|v_g|t<\ell} dk\, |v_g|\, s(k) + \ell \int_{2|v_g|t>\ell} dk\, s(k)}$$

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- Def's in LHL:

$$v_g = \frac{d\omega(k)}{dk}$$

$$n(k) := \langle a_k^\dagger a_k \rangle = \frac{1}{4}\left(\frac{\omega(k)}{\omega_0(k)} + \frac{\omega_0(k)}{\omega(k)}\right) - \frac{1}{2},$$

$$s(k) = \frac{1}{2\pi}\left[(n(k)+1)\log(n(k)+1) - n(k)\log n(k)\right]$$

Alba-Calabrese Picture for $m \neq 0$

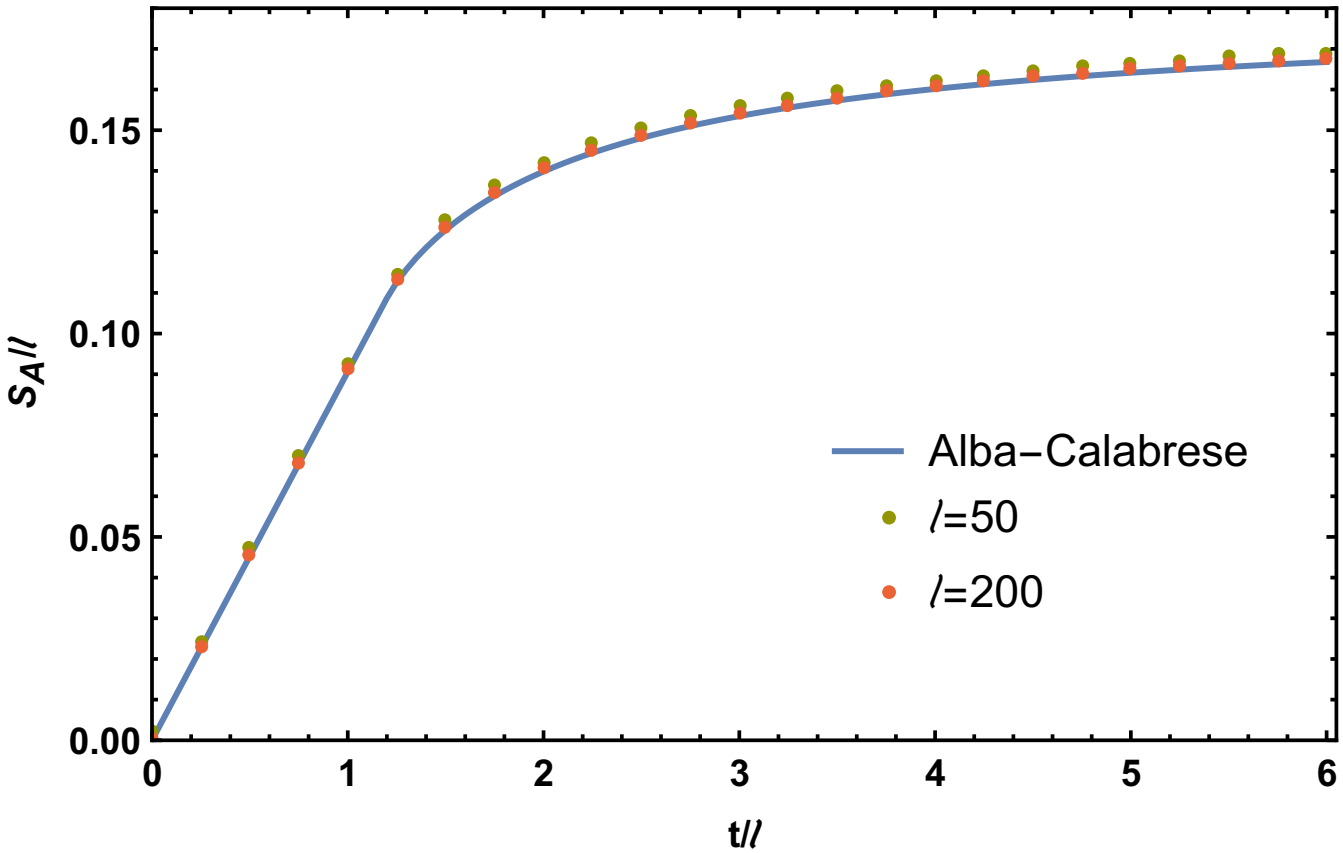

Figure: Infinite lattice with $m_0 = 1$ and $m = 2$

Alba-Calabrese Picture for $m \to 0$

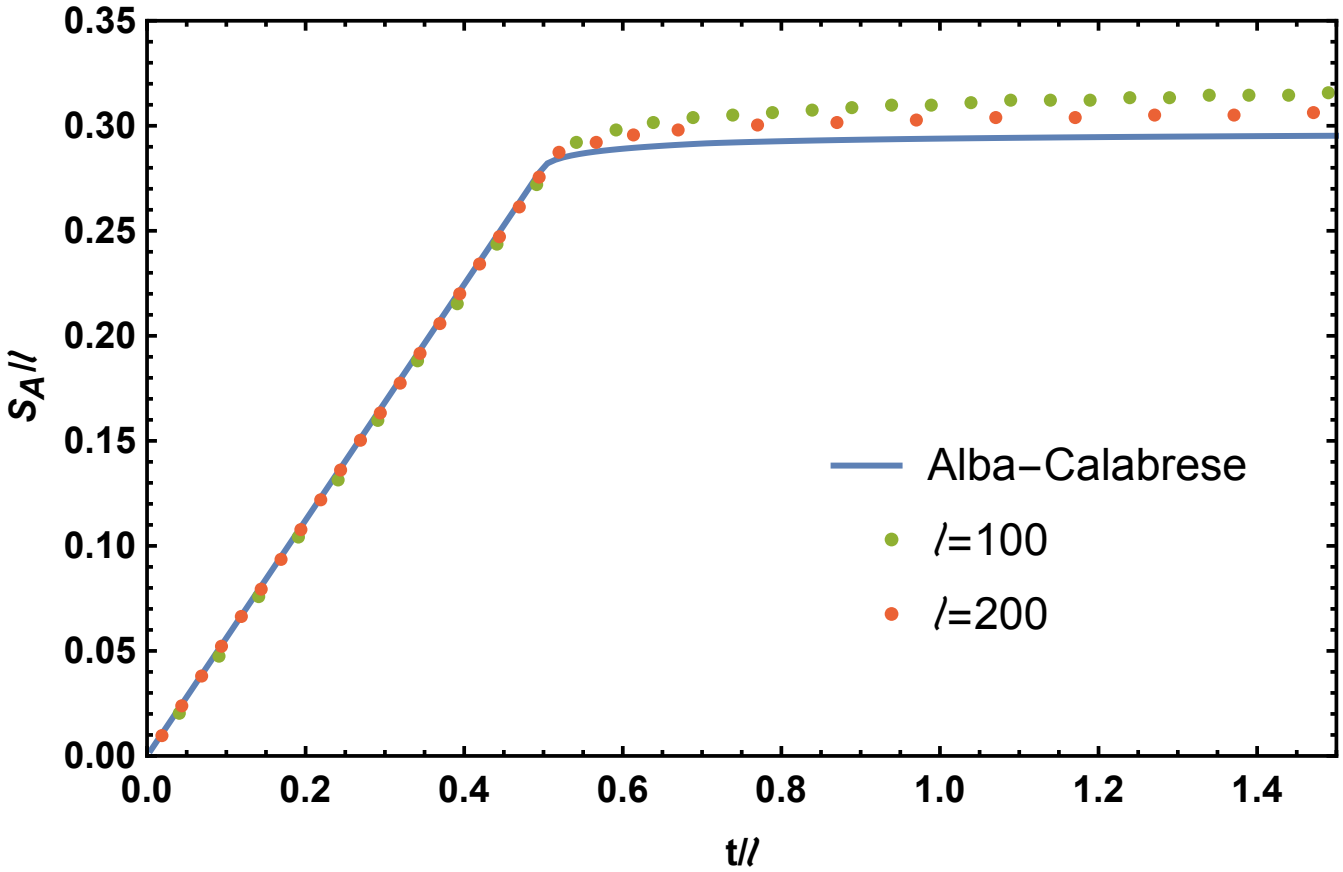

Figure: Infinite lattice with $m_0 = 1$ and $m = 10^{-6}$

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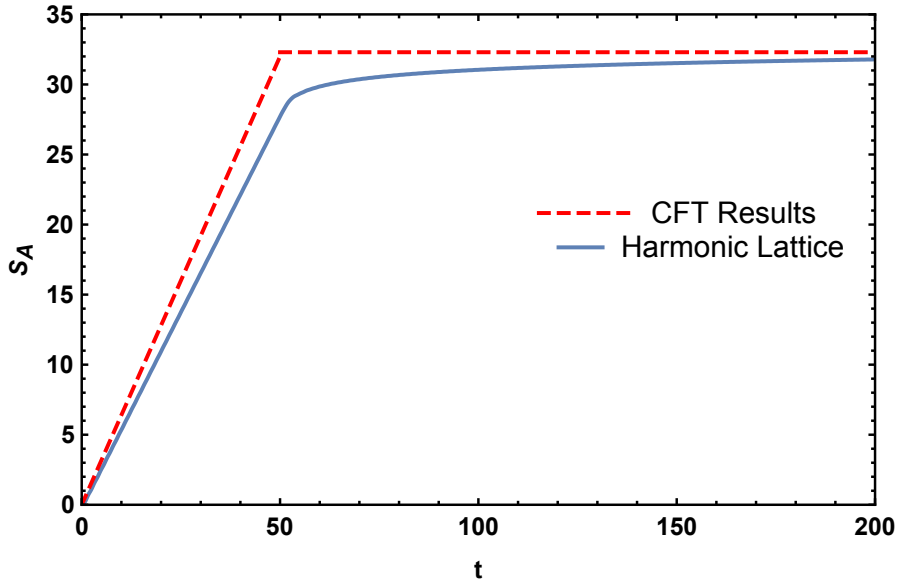

Effective Saturation

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- Analytical behavior after effective saturation

$$\mathcal{S}(t > \ell/2) = \mathcal{S}_{t=\ell/2} + \log \frac{t}{\ell}$$

[Chapman, Eisert, Hackl, Heller, Jefferson, Marrochio, Myers '18]

Ent. Propagation in LQFTs

Quasi-Particle Picture

- The post and pre quench dispersion relation

$$\omega_k = \sqrt{m^{2z} + \left(2\sin\frac{k}{2}\right)^{2z}}, \qquad \omega_{0,k} = \sqrt{m_0^{2z} + \left(2\sin\frac{k}{2}\right)^{2z}}$$

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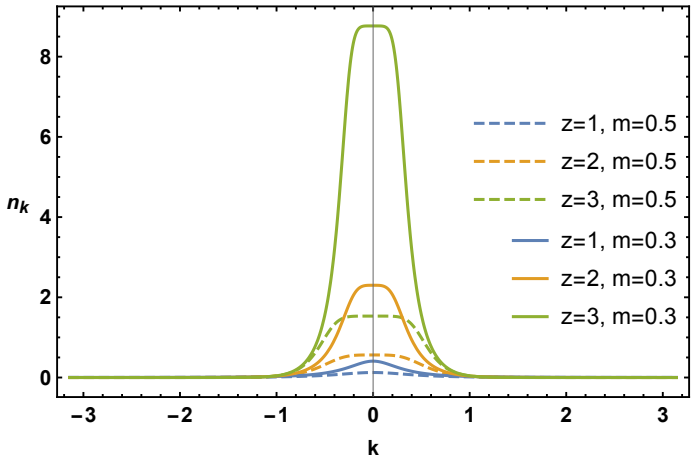
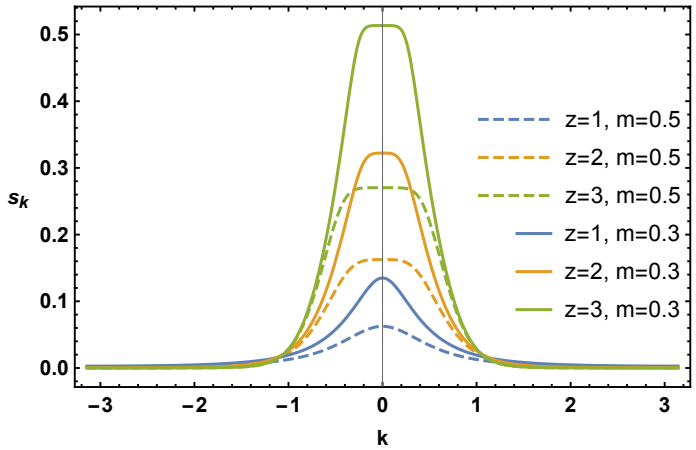
- Max. group velocity (in $m \to 0$ limit)

$$v_g^{\max}(z) = 2^{z-1}\sqrt{z}\left(\frac{z-1}{z}\right)^{\frac{z-1}{2}} + \mathcal{O}\left(m^{4z}\right)$$

$k \to 0$ Behaviour

- Divergence of $n(k)$ and $s(k)$ as $k \to 0$





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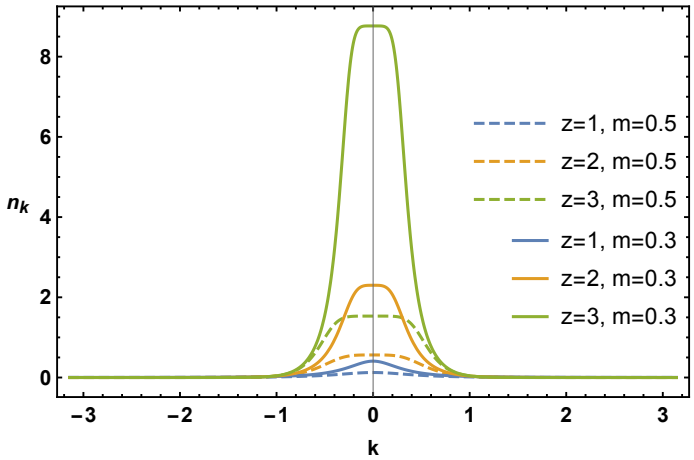

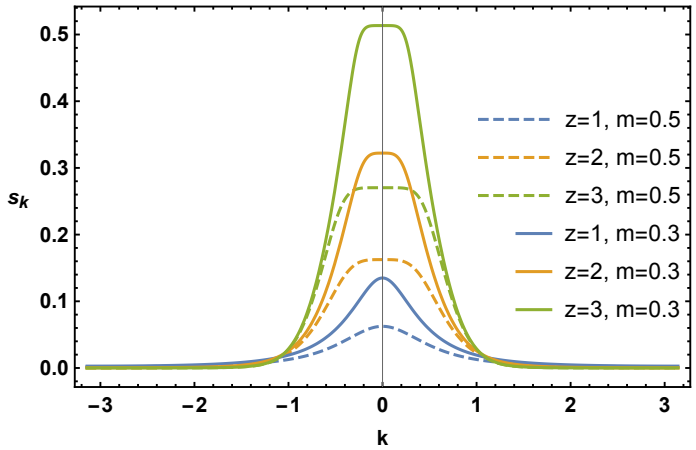

- Occupation number

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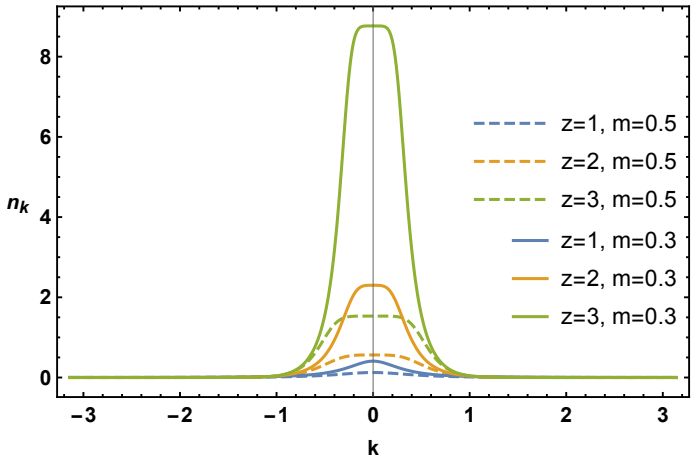

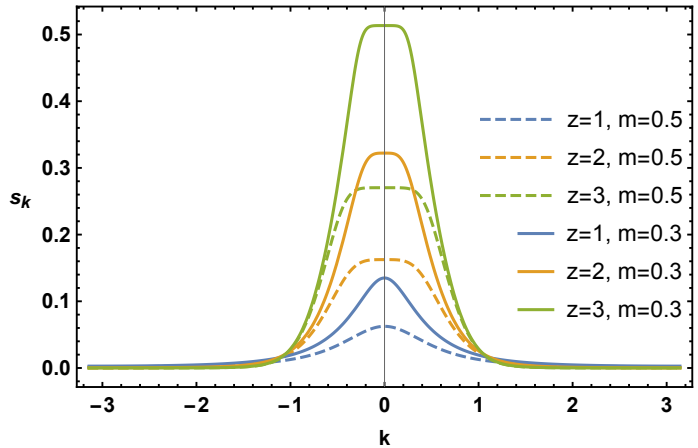

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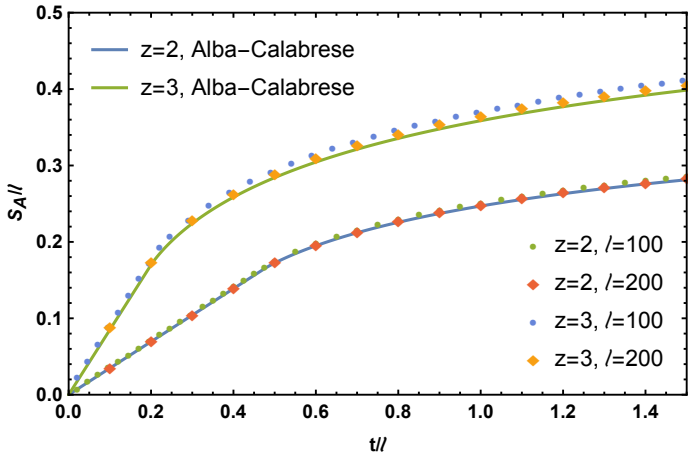
- $n(k)$ is divergent if

$$\frac{m_0}{m} \ll 1 \quad , \quad \frac{m}{m_0} \ll 1 \quad , \quad z \gg 1$$

How good QP picture is?

- Case $m \neq 0$



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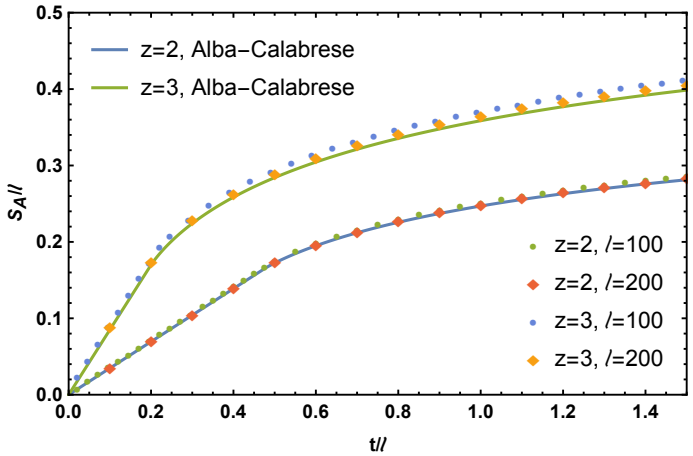

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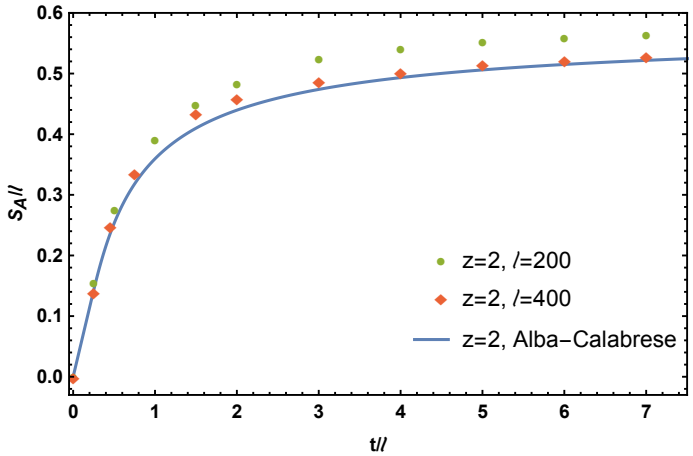
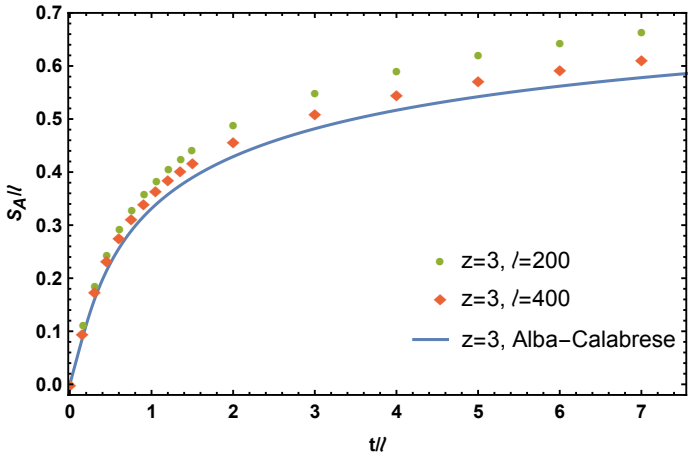
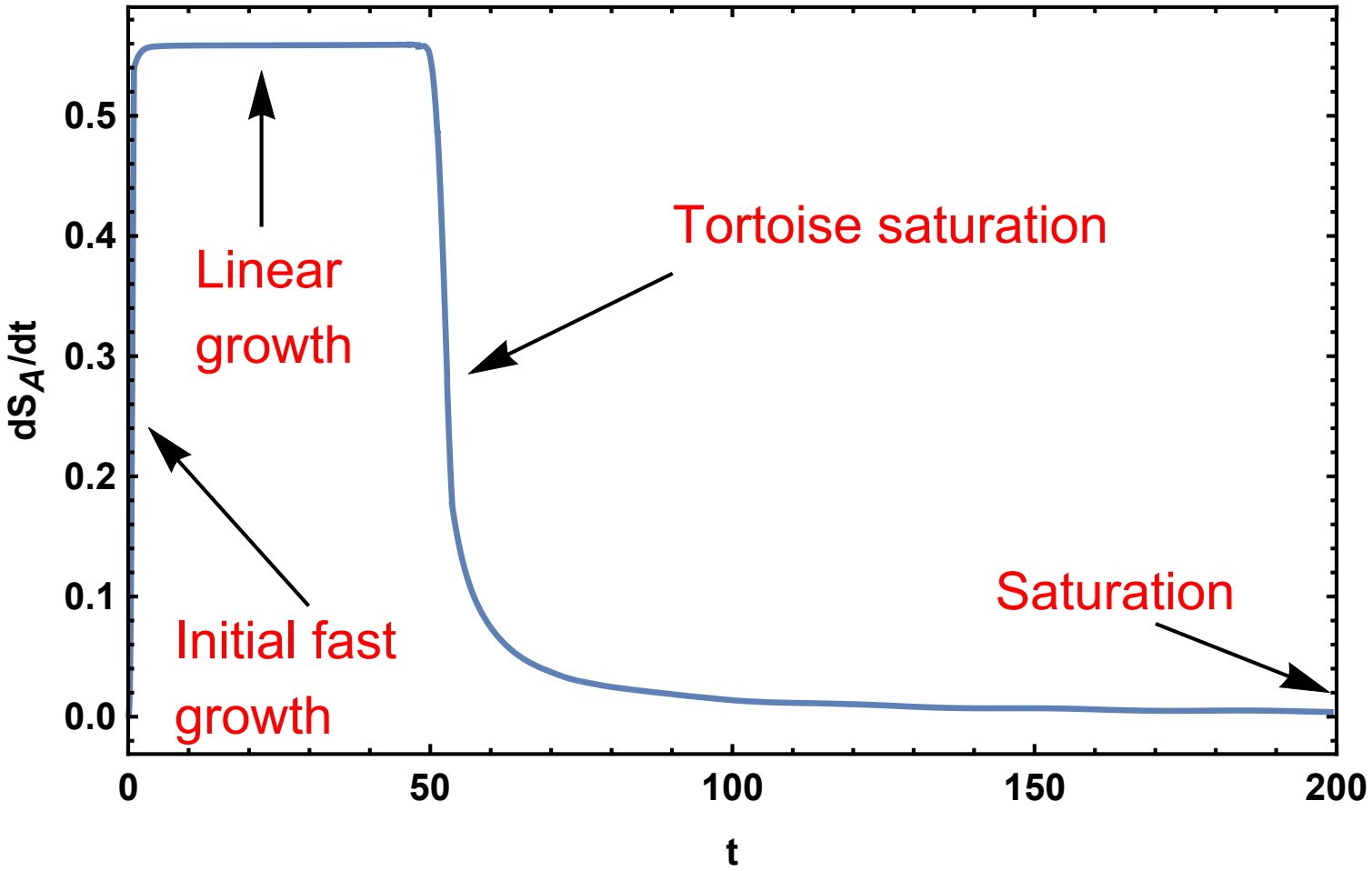




dS/dt for $z = 1$


0.5
0.4
0.3
0.2
0.1
0.0
dS_A/dt
0
50
100
150
200
t
Linear growth
Tortoise saturation
Initial fast growth
Saturation

Numerical Results: Effective Saturation for $z = 1$

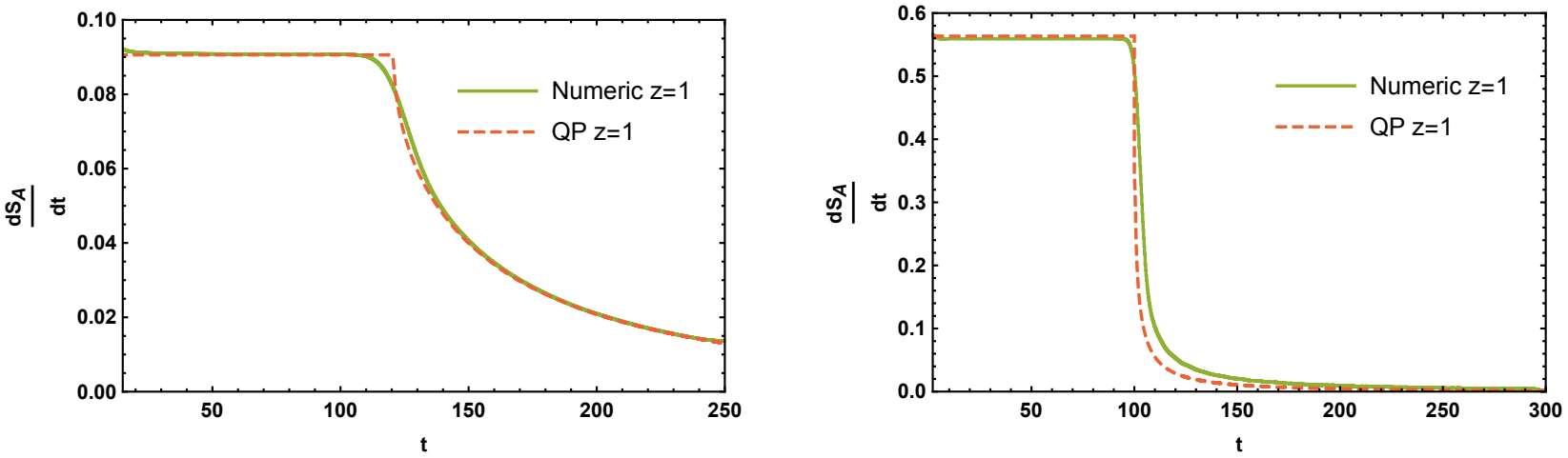


Figure: Left: $m = 2$, Right: $m = 10^{-6}$

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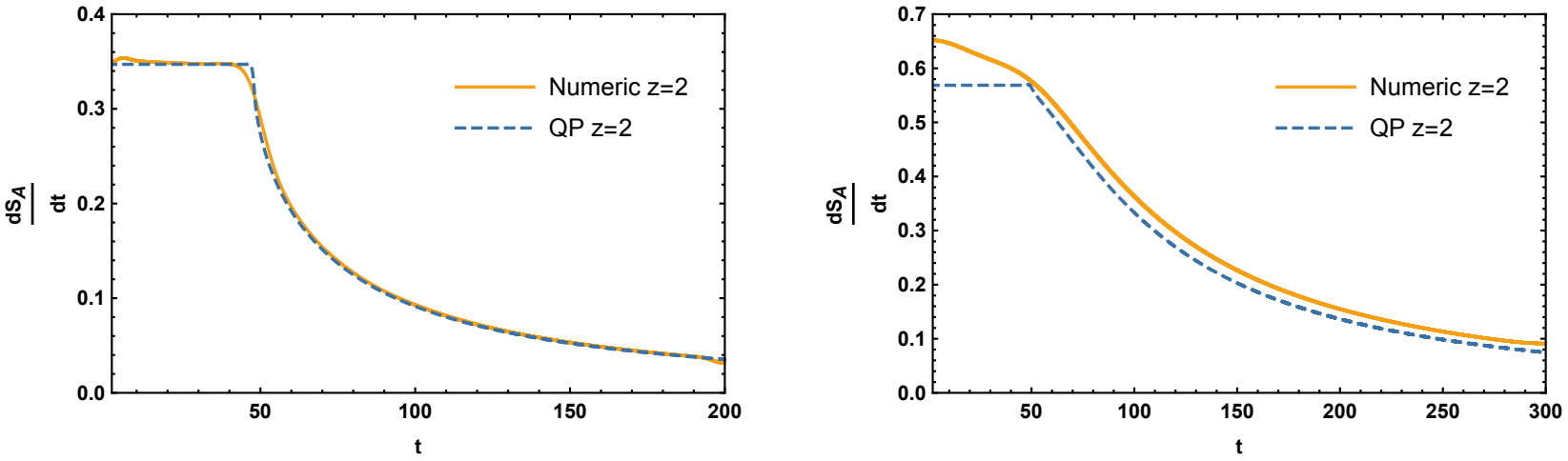

Figure: Left: $m = 2$, Right: $m = 10^{-6}$

- for $(t >, t_s^{\max}(2),\ m_0 \gg 1,\ m \ll 1)$

$$\mathcal{S}(t) = \mathcal{S}_0 + \frac{1}{t}\left(c_1 + c_2 \log t\right) m + \mathcal{O}\left(m^2\right)$$

Numerical Results: Effective Saturation for $z = 3$

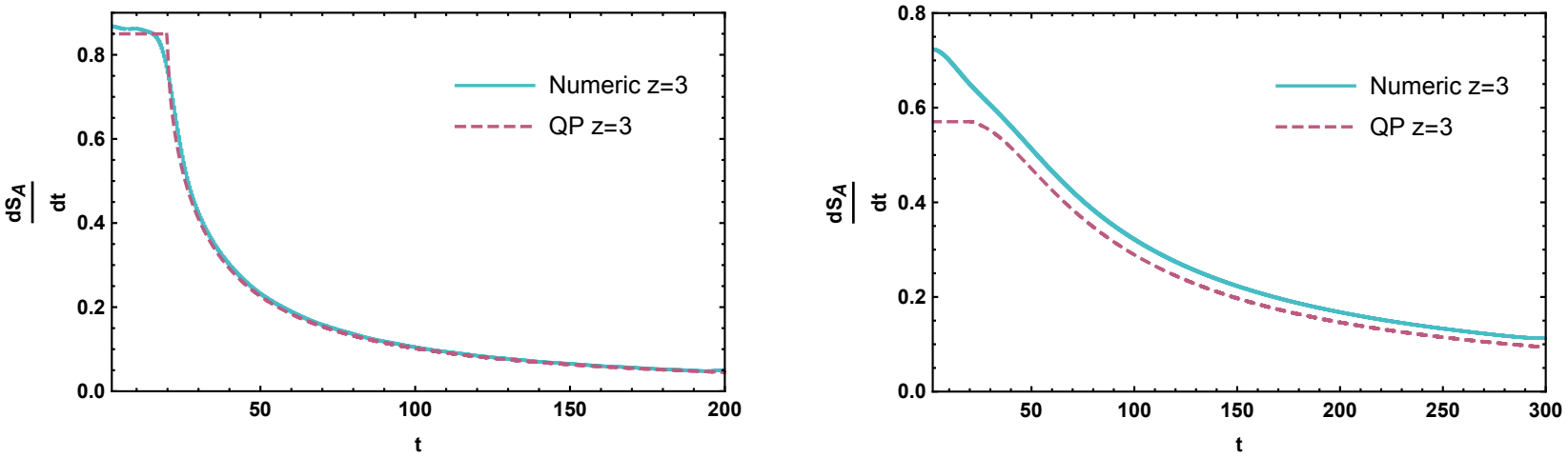

Figure: Left: $m = 2$, Right: $m = 10^{-6}$

Widen Light-cones

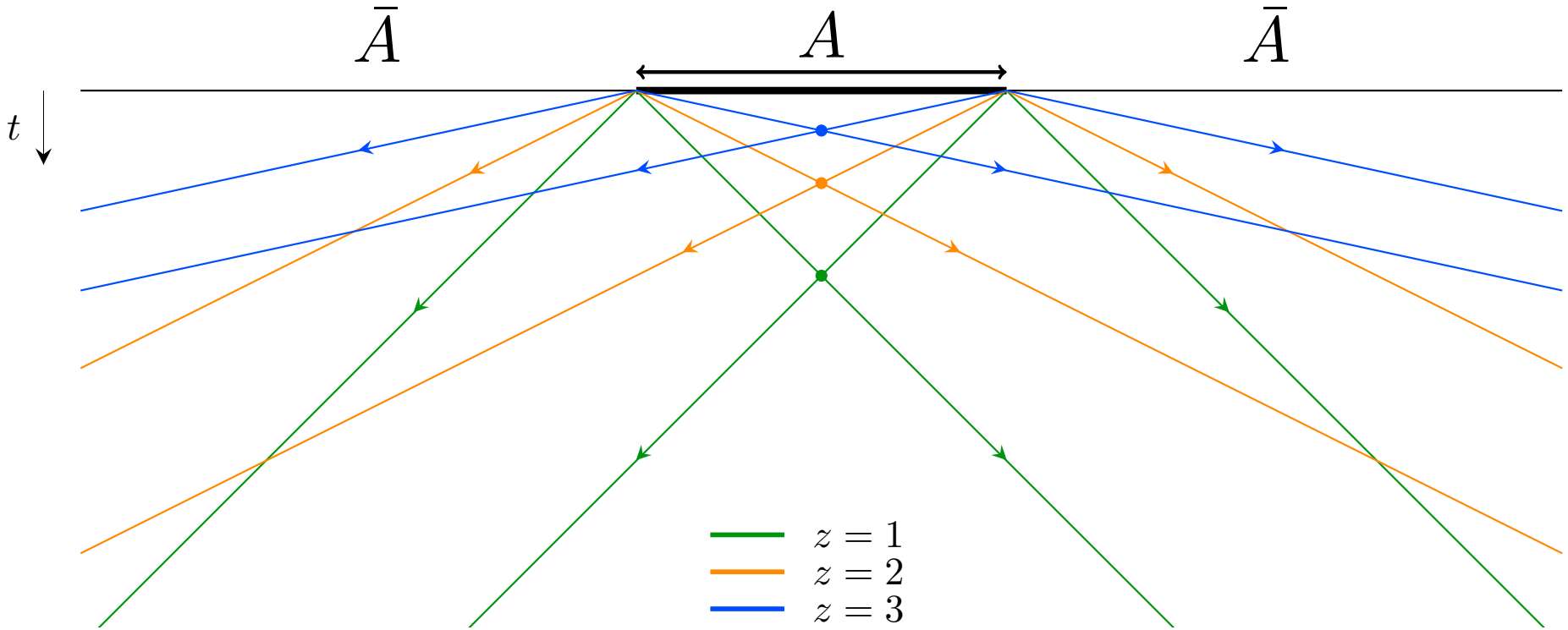

Figure: $\{t_s^{\max}(1), t_s^{\max}(2), t_s^{\max}(3)\}$

Continuum versus Lattice

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- lack of boost symmetry

  $\Rightarrow$ it does not vanish for spacelike points

Continuum versus Lattice

- Lattice version of the theory hints to similar physics in the continuum

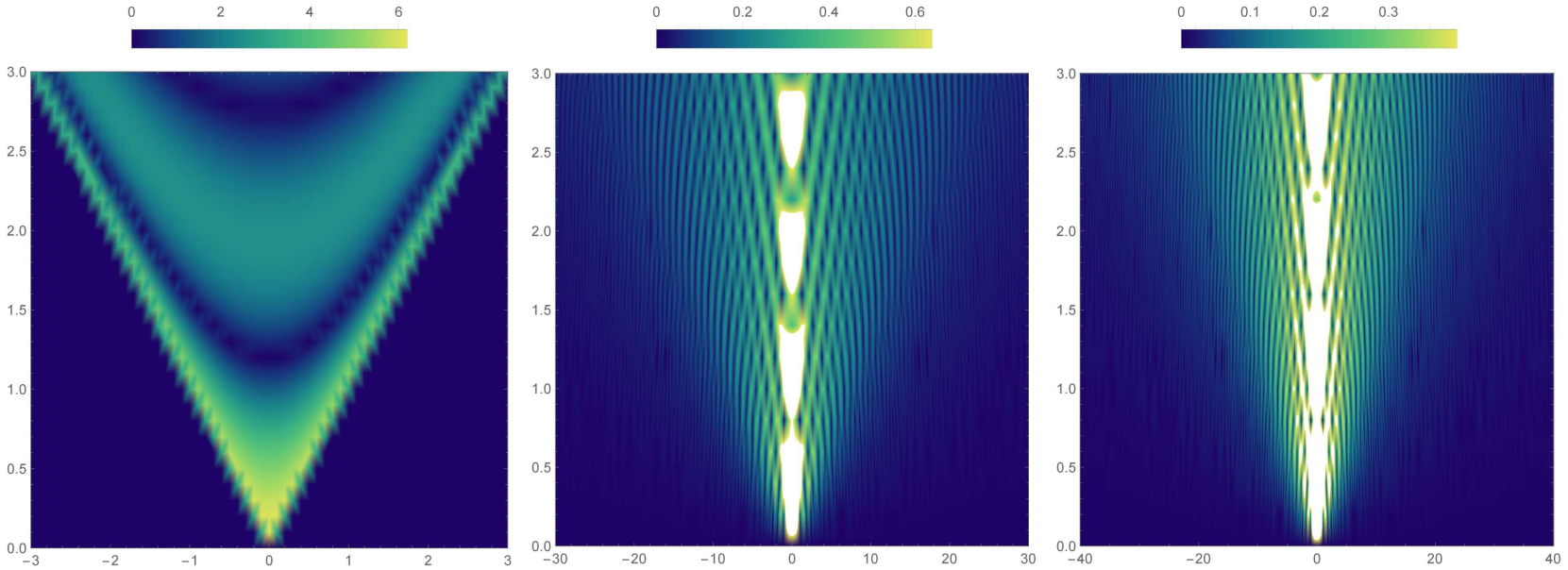

Figure: $m = 2$ and $z = 1, 2, 3$

Summary / Further Directions

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  - EE is increased w/ z and z^2 in different regimes
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