Conformal Bootstrap in percolation and related models

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October 25, 2016

Plan

- Introduction
- CFT and bootstrap, general results.
- Potts models in two dimensions.
- Conformal bootstrap for Potts models.
- Conclusion

Based on a work done with

Sylvain Ribault, IPhT, CEA/Saclay, France

and

Raoul Santachiara, LPTMS, Université Paris-Saclay

A first part is available as
"**A conformal bootstrap approach to critical percolation in two dimensions**", Marco Picco, Sylvain Ribault, Raoul Santachiara
SciPost Phys., 1(1), 009 (2016) and http://arxiv.org/abs/1607.07224

Introduction

Introduction

- Bootstrap program : Solve the CFT from consistency conditions without assuming the Lagrangian.
  Ferrara, Gatto and Grillo (1973) and Polyakov (1974).
- Very ambitious since there is an infinite number of unknown parameters and also an infinite number of constraints.
- CFT developed much more late in the 80's by using some additional conditions (unitarity + minimality). $\rightarrow$ BPZ
- Recently, many progress in $3d$ (and $> 3d$) for the Ising model with a bootstrap approach.
- I will present results for the Potts model in $2d$. For some values of Q, it corresponds to a unitary minimal model CFT. For general values of Q, it is not. General case and in particular the limit $Q \rightarrow 1$, i.e. percolation ?

CFT and bootstrap, general results

CFT and bootstrap

A CFT is specified by :

- Spectrum $\mathcal{S} = \{\mathcal{O}_i\}$ of primary operators with their dimensions and spins : Δ_i, s_i.
- The OPE : operator product expansion for primary operators :

$$\mathcal{O}_i(x)\mathcal{O}_j(0) \simeq \sum_k C_{ij}^k P(x, \partial_x)\mathcal{O}_k(x) \quad (1)$$

  with $P(x, \partial_x)$ describing the descendants.

- Δ_i, s_i, C_{ij}^k describe the CFT, *i.e.* we can compute any correlation function.
- The problem is to determine consistent sets of such data.

CFT and bootstrap

Conformal invariance (in D dimensions) :

- Translation by a:

$$x \to x + a \tag{2}$$

- Dilatation by λ:

$$x \to \lambda x \tag{3}$$

- Special conformal transformations (SCT) by y:

$$x \to \frac{x + xy^2}{1 + 2x.y + x^2 y^2} \tag{4}$$

CFT and bootstrap

A first step is to use the symmetries corresponding to the conformal invariance : Conformal kinematics

- two-point correlation function is fixed by conformal symmetry (dilatation) :

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}} \tag{5}$$

  This also fixes the normalisation of the fields $\mathcal{O}_i$.

- three-point correlation function is also constrained by the conformal symmetry (special conformal transformations) :

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle \simeq \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|y-z|^{\Delta_j+\Delta_k-\Delta_i}|x-z|^{\Delta_i+\Delta_k-\Delta_j}} \tag{6}$$

  and the C_{ijk} are the same which already appeared in the OPE.

CFT and bootstrap

Next we consider the four-point correlation function. This is more complicated

- Starting from a general function $f(z_1, z_2, z_3, z_4)$ we can impose $z_1 = 0$ (translation) : $f(0, z_2, z_3, z_4)$
- Next we use the SCT to impose $z_2 \to \infty$: $f(0, \infty, z_3, z_4)$
- Next we use rotation + dilatation to put z_3 at a fixed point : $f(0, \infty, 1, z_4)$.

That's it. z_4 can not be fixed ! So the four point correlation function will depend on one variable.

CFT and bootstrap

Thus from the kinematics, we can only get :

$$\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\mathcal{O}_k(z_3)\mathcal{O}_l(z_4)\rangle = \left(\frac{|z_{24}|}{|z_{14}|}\right)^{\Delta_i-\Delta_j}\left(\frac{|z_{14}|}{|z_{13}|}\right)^{\Delta_k-\Delta_l}\frac{g(u,v)}{|z_{12}|^{\Delta_i+\Delta_j}|z_{34}|^{\Delta_k+\Delta_l}} \tag{7}$$

with $g(u,v)$ an arbitrary function of the conformally invariant cross-ratios

$$u = \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \;\; ; \;\; v = \frac{z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2} \tag{8}$$

and $z_{12} = z_1 - z_2$, etc.

CFT and bootstrap

We will consider from now on the four-point correlation function of the spin operator, with an associated field ϕ_σ and a dimension Δ_σ. This makes the task much easier since then we can also use the invariance under permutations :

$$\begin{aligned}\langle\phi_\sigma(z_1)\phi_\sigma(z_2)\phi_\sigma(z_3)\phi_\sigma(z_4)\rangle &= \langle\phi_\sigma(z_2)\phi_\sigma(z_3)\phi_\sigma(z_4)\phi_\sigma(z_1)\rangle \\ &= \langle\phi_\sigma(z_3)\phi_\sigma(z_4)\phi_\sigma(z_1)\phi_\sigma(z_2)\rangle \\ &= etc.\end{aligned}$$

Next, we use the OPE for performing an expansion of the four-point correlation function

$$\phi_\sigma(z_1)\phi_\sigma(z_2) \simeq \sum_k C^k_{\sigma\sigma}\mathcal{O}_k \; ; \; \phi_\sigma(z_3)\phi_\sigma(z_4) \simeq \sum_{k'} C^{k'}_{\sigma\sigma}\mathcal{O}_{k'} \tag{9}$$

CFT and bootstrap

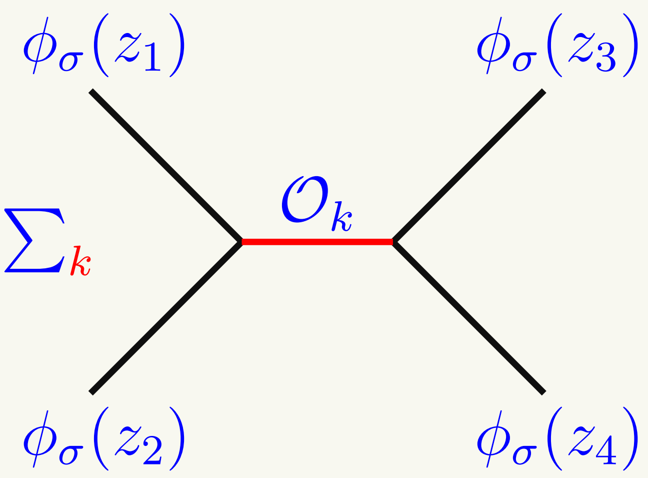
Putting all this in the four-point correlation function, one gets

$$\langle \phi_\sigma(z_1)\phi_\sigma(z_2)\phi_\sigma(z_3)\phi_\sigma(z_4)\rangle = \sum_k (C^k_{\sigma\sigma})^2 \langle \mathcal{O}_k \mathcal{O}_k \rangle \tag{10}$$

This is represented as

$$\langle \phi_\sigma(z_1)\phi_\sigma(z_2)\phi_\sigma(z_3)\phi_\sigma(z_4)\rangle = \left\langle \phi_\sigma(z_1)\phi_\sigma(z_2)\phi_\sigma(z_3)\phi_\sigma(z_4) \right\rangle \tag{11}$$

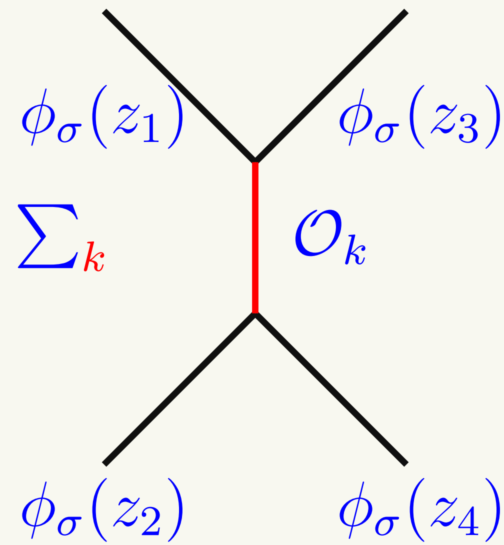
$= \sum_k$



We could have started after a permutation. Then we would get

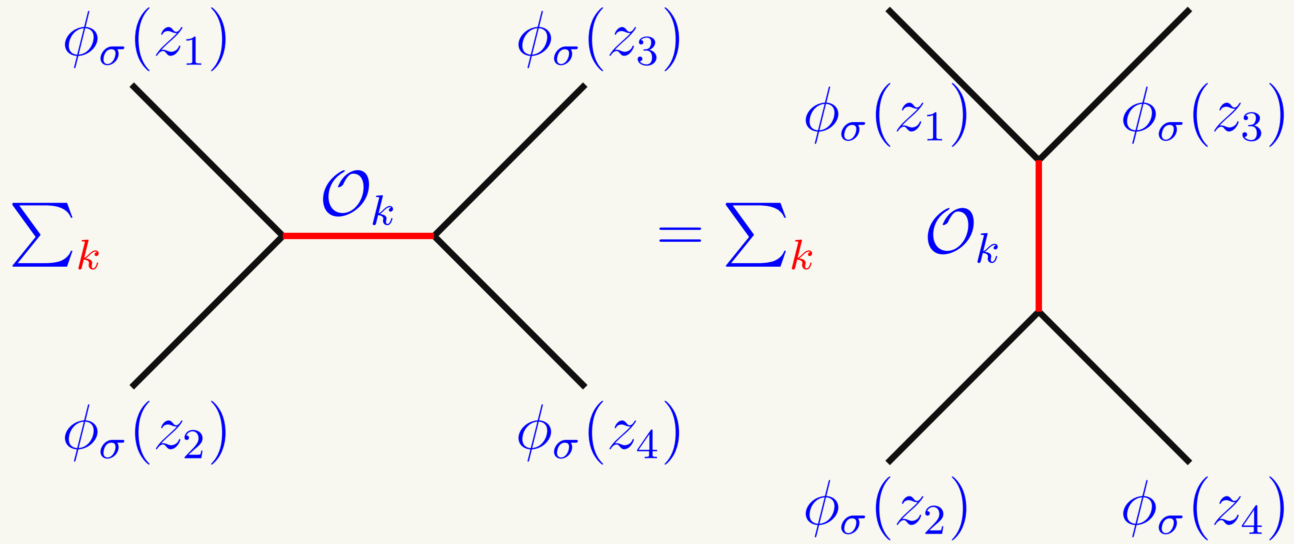
$$\langle \phi_\sigma(z_1)\phi_\sigma(z_2)\phi_\sigma(z_3)\phi_\sigma(z_4)\rangle = \left\langle \phi_\sigma(z_1)\phi_\sigma(z_2)\phi_\sigma(z_3)\phi_\sigma(z_4) \right\rangle \tag{12}$$

$$= \sum_k$$



CFT and bootstrap

Graphically, this gives the conformal bootstrap equation



This corresponds to the s channel and t channel. There is also a third channel (u) : $\phi_\sigma(z_1)\phi_\sigma(z_2)\phi_\sigma(z_3)\phi_\sigma(z_4)$

CFT and bootstrap

In more details, the $s - t$ conformal bootstrap equation is

$$\sum_k (C^k_{\sigma\sigma})^2 \mathcal{F}^{z_1 z_2, z_3 z_4}_{\Delta_k, s_k}(u, v) = \sum_k (C^k_{\sigma\sigma})^2 \mathcal{F}^{z_1 z_3, z_2 z_4}_{\Delta_k, s_k}(v, u) \tag{13}$$

with $\mathcal{F}^{z_1 z_2, z_3 z_4}_{\Delta_k, s_k}(u, v)$ the conformal block for the operator k with dimension Δ_k and spin s_k and

$$u = \frac{z_{12} z_{34}}{z_{13} z_{24}} \quad ; \quad v = \frac{z_{14} z_{23}}{z_{13} z_{24}} \tag{14}$$

and $z_{12} = z_1 - z_2$, etc.
Each conformal block $\mathcal{F}^{z_1 z_2, z_3 z_4}_{\Delta_k, s_k}(u, v)$ corresponds to one operator $\mathcal{O}_k$ in the OPE with descendants.

CFT and bootstrap

- To solve these equations, one considers

$$\sum_k (C^k_{\sigma\sigma})^2 (\mathcal{F}^{z_1 z_2, z_3 z_4}_{\Delta_k, s_k}(u,v) - \mathcal{F}^{z_1 z_3, z_2 z_4}_{\Delta_k, s_k}(v,u)) =$$

$$\sum_k p_k f_k = 0 \qquad (15)$$

with $p_k > 0$.

- One then searches for conditions on Δ_k, s_k such that the set of f_k spans (or not) a positive cone.
- If it spans a positive cone, then eq.(15) can not be satisfied.
- This gives restrictions on the values of Δ_k, s_k.
- Further details : S. Rychkov, arXiv:1601.05000

Potts models in two dimensions

Potts models in two dimensions

- Potts model is a simple spin model.
- On a regular lattice $\mathcal{G}$, each site contains a variable σ_i which can takes one among Q values and the contribution to the energy for two neighboring sites is $\delta_{\sigma_i,\sigma_j}$.

$$\mathcal{H} = -\sum_{<ij>} \delta_{\sigma_i,\sigma_j} \quad ; \quad Z = \sum_{\sigma_i} e^{-\beta\mathcal{H}} \tag{16}$$

- This model can be considered in any dimension.
- In two dimensions, the model is critical for $\beta = \log\left(1+\sqrt{Q}\right)$. For $Q \leq 4$ it corresponds to a second order phase transition *i.e.* a CFT.

Potts models in two dimensions

Some well know models are :

- $Q = 2$: Ising model, $m = 3$ CFT with $c = 1/2$,
- $Q = 3$: 3-state Potts model, $m = 5$ CFT with $c = 0.8$
- $Q = 4$: 4-state Potts model, $m \to +\infty$ CFT with $c = 1.0$
- Other values of m : tricritical Potts model : Potts model with vacancies or dilution.
- Extension for non integer values of Q: Potts random cluster model.

Potts models in two dimensions

- We consider graphs $\mathcal{G}$ build by adding randomly bonds with a probability p. Each graph contributes with the following probability :

$$\text{Probability}(\mathcal{G}) = Q^{\#\,\text{clusters}} p^{\#\,\text{bonds}} (1-p)^{\#\,\text{edges without bond}} \quad (17)$$

This model is critical for each value of $Q \leq 4$ and for

$$p_c = \frac{\sqrt{Q}}{\sqrt{Q}+1}$$

- At $p = p_c$, local conformal invariance.
- $p_c = 1 - e^{-\beta} \to$: Fortuin-Kasteleyn clusters.
- $Q = 4\cos^2 \pi\gamma^2, c = 1 - 6(\gamma - \frac{1}{\gamma})^2, (|\gamma| \leq 1)$
- For $Q = 2, 3, 4$ corresponds the spin models $\sigma_i = 1, \cdots, Q$.

Potts models in two dimensions

Computation of correlation function is done by the construction of random cluster :

- $\langle\sigma(z_1)\sigma(z_2)\rangle = \text{Probability}(z_1 \sim z_2)$ with $\text{Probability}(z_1 \sim z_2)$ the probability that z_1 and z_2 are in the same random cluster.
- $\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\rangle = \text{Probability}(z_1 \sim z_2 \sim z_3)$. Delfino, Viti (2010) and Delfino, Picco, Santachiara and Viti (2013).
- $\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = \text{Probability}(z_1 \sim z_2 \sim z_3 \sim z_4)$? More complicated than that. For that case, we can have factorizations : $(z_1 \sim z_2)$ and $(z_3 \sim z_4)$ or $(z_1 \sim z_3)$ and $(z_2 \sim z_4)$, etc, which can also contribute.
  For example, for $Q = 2$ with $\sigma = \pm 1$, $\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle$ will get a contribution from $\sigma(z_1) = 1$, $\sigma(z_2) = 1$, $\sigma(z_3) = 1$, $\sigma(z_4) = 1$ which will be the same as from $\sigma(z_1) = 1$, $\sigma(z_2) = 1$, $\sigma(z_3) = -1$, $\sigma(z_4) = -1$

Potts models in two dimensions

- In general, we have to consider the four types of clusters (Delfino and Viti, 2010, 2011)
  $P_0 : z_1 \sim z_2 \sim z_3 \sim z_4$; $P_1 : z_1 \sim z_2$ and $z_3 \sim z_4$
  $P_2 : z_1 \sim z_4$ and $z_2 \sim z_3$; $P_3 : z_1 \sim z_3$ and $z_2 \sim z_4$
- For the Q Potts models, the ”ordinary” four point correlation function is

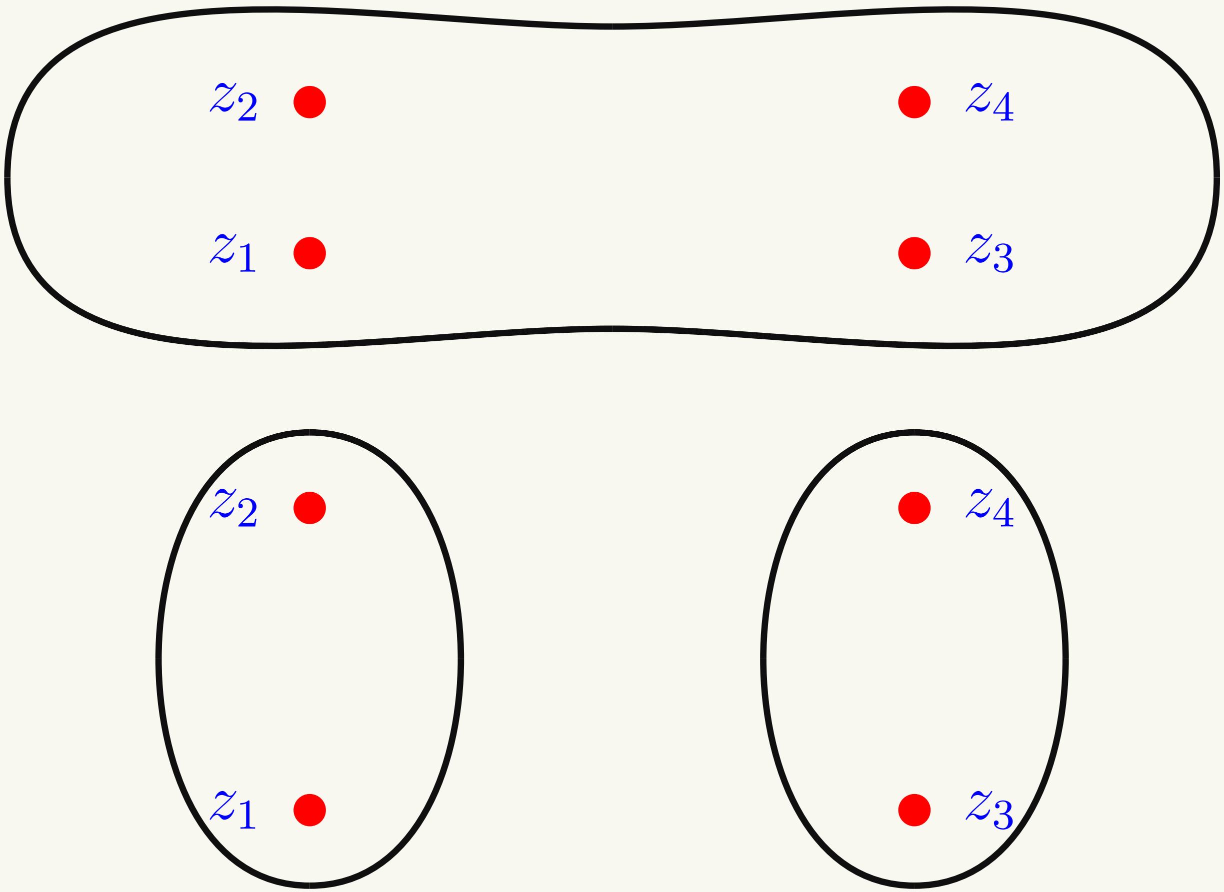
$$\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle \simeq P_0 + \frac{(Q-1)}{Q^2-3Q+3}(P_1+P_2+P_3) \quad (18)$$

up to some normalization (see later).

Potts models in two dimensions

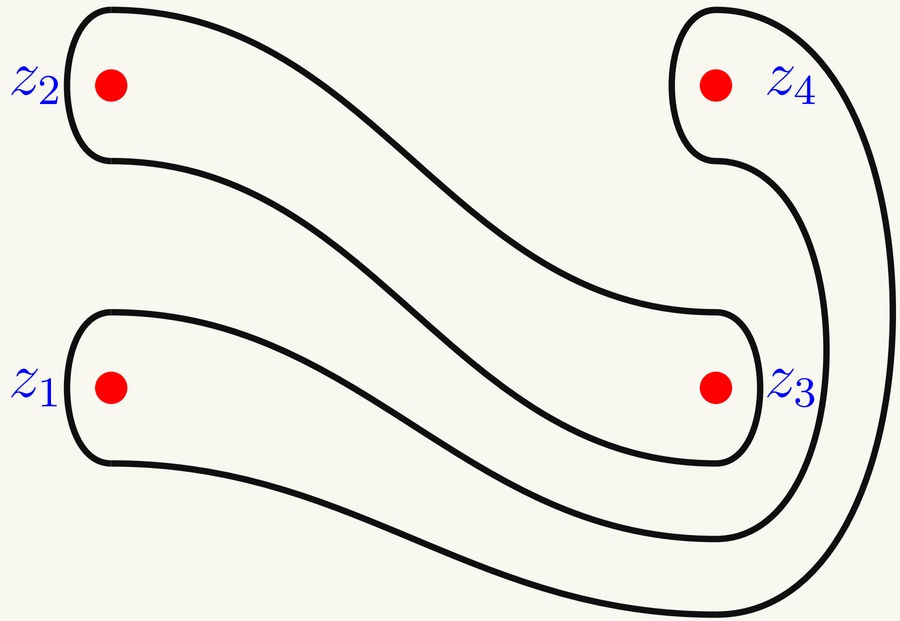
P_0 :

P_1 :

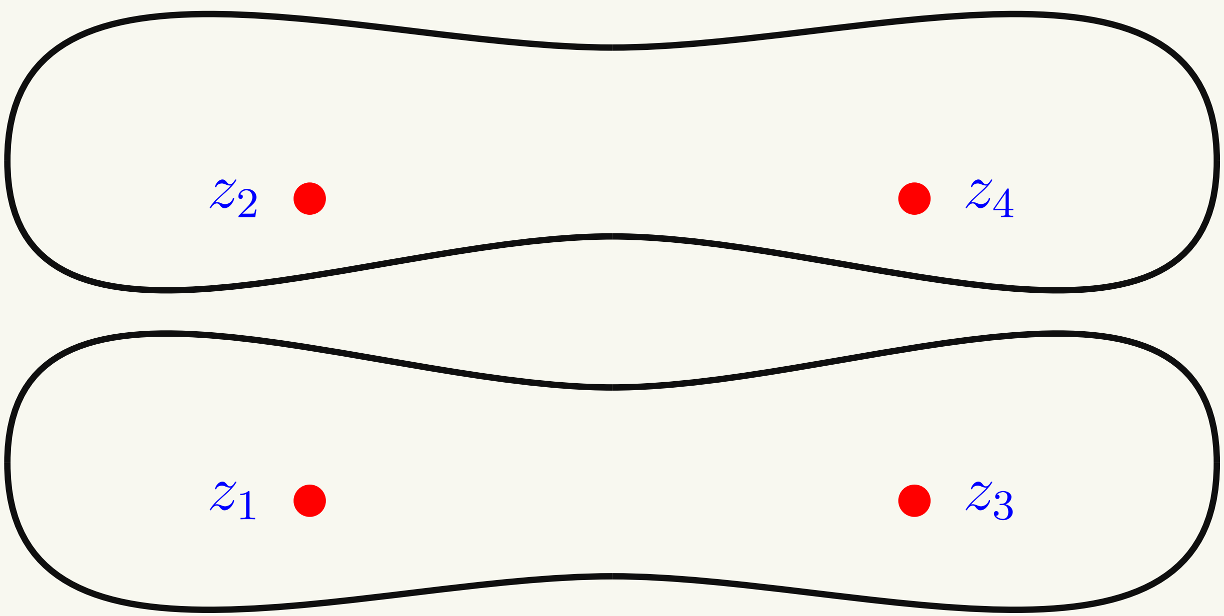


Potts models in two dimensions

P_2 :



P_3 :



Potts models in two dimensions

We can consider the special limit with $z_1 \to z_2$ and $z_3 \to z_4$ while z_1 is very far from z_3.

- P_0 will go to the limit $A_2(z_1, z_2)A_2(z_3, z_4) \times A_2(z_1, z_3)$
  So in the limit $|z_1 - z_2| = |z_3 - z_4| = a$ (the lattice spacing), we will get

  $$P_0(z) \simeq A_2^2 z^{-2\Delta} \tag{19}$$

  with A_2 the normalization of the two point correlation function (and $z = |z_1 - z_3|$. Different from the z used later on).
- In the same limit, we then expect

  $$P_1(z) \simeq A_2^2(1 - z^{-2\Delta}) \tag{20}$$

  The first term can be interpreted like the presence of the identity operator.

Potts models in two dimensions

- Difficult to predict for $P_2(z)$. But it will be very small !!!
- For $P_3(z)$, we also expect a small exponent since in the limit of small distance $z_1 - z_2$ and $z_3 - z_4$, the two clusters must not touch. One can expect that it is related to the some boundary operator.
- Note also that in this limit, we can also fixe the normalization (on the lattice) of eq.(20) :

$$\begin{aligned}\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle &= \mathcal{N}(P_0 + \frac{(Q-1)}{Q^2-3Q+3}(P_1+P_2+P_3)) \\ &= A_2^2 \end{aligned} \quad (21)$$

and then the normalization on the lattice is chosen such that (Identity is normalized to one in CFT!).

$$\mathcal{N} = \frac{Q^2-3Q+3}{Q-1} \times \frac{1}{A_2^2} \quad (22)$$

Potts models in two dimensions

First Numerical results

- We expect the general behaviour (after changing the normalisation)

$$\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle \simeq |z_{12}|^{-4\Delta_\sigma}|z_{34}|^{-4\Delta_\sigma} \left(1+\sum_i z^{2\Delta_i}F_i(z)+\cdots\right) \quad (23)$$

  with $z=\frac{z_{12}z_{34}}{z_{13}z_{24}}$. Δ_σ is the conformal dimension (the physical dimension is $2\Delta_\sigma = 1/8$ for Ising).

- In the following, we always remove the trivial part $|z_{12}|^{-4\Delta_\sigma}|z_{34}|^{-4\Delta_\sigma}$.
- We measure separately P_0, P_1, P_2, P_3. We then fit them to the previous form looking for the smallest values for Δ_i.

Potts models in two dimensions

- We obtain, for all values of Q,

$$P_0 \simeq \alpha_0 z^{2\Delta_\sigma}(1+\cdots) + z^{\Delta_2} \tag{24}$$

  and Δ_2 is large. In particular, there is no identity. This is in agreement with the previous arguments.

- Again, for all values of Q, we obtain

$$P_1 \simeq 1 - \alpha_0 z^{2\Delta_\sigma}(1+\cdots) + \cdots \tag{25}$$

  Here, we obtain the same constant α_0 !!! This is also in agreement with the previous arguments.

- P_2 is small, difficult to fit
- $P_3 \simeq r^{2\Delta_i}(1+\cdots)$ with $2\Delta_i \simeq 1.5$ large.

Potts models in two dimensions

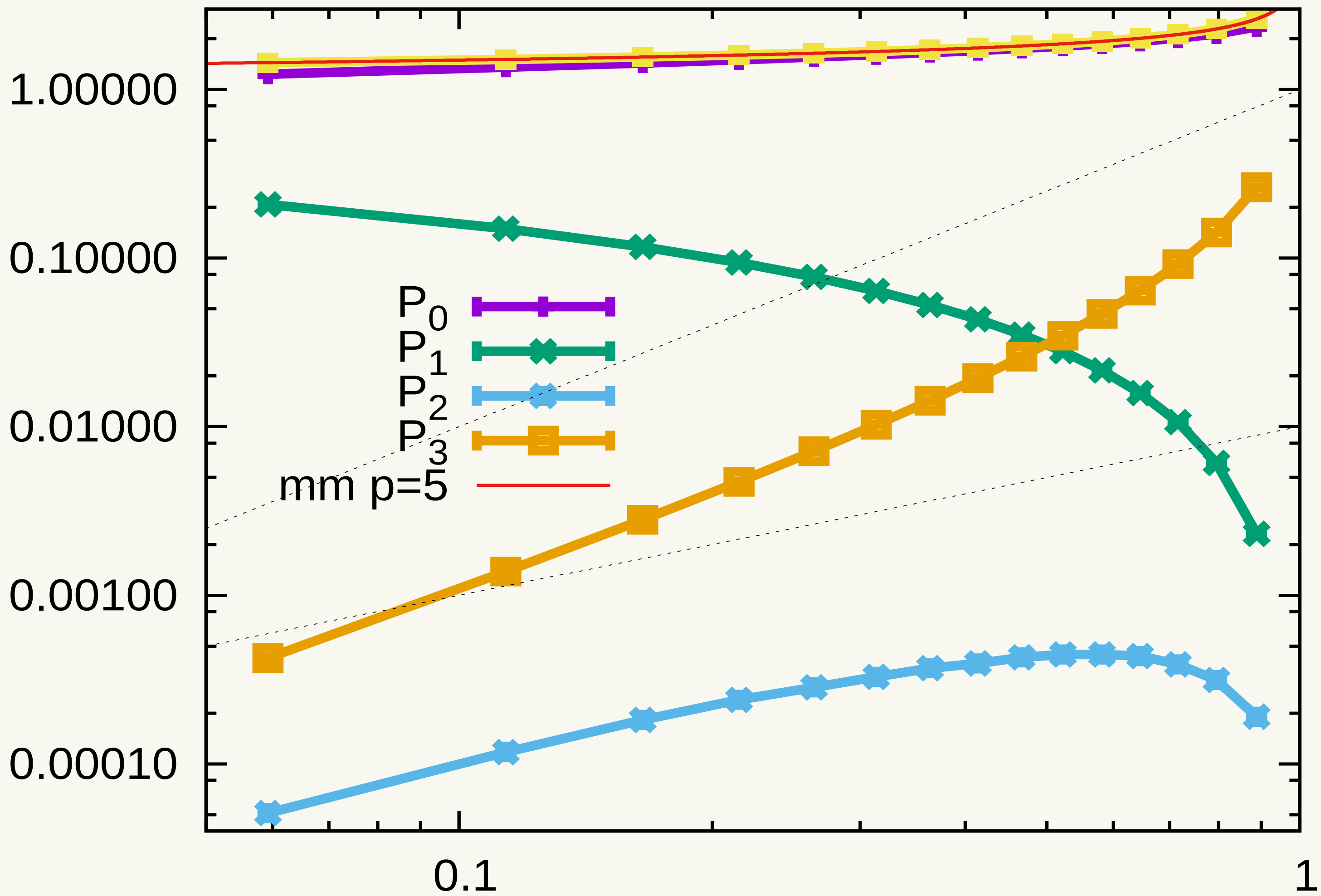


Figure 1: Data for $Q = 3$ Potts model

Potts models in two dimensions

- We can compare with simple models: Ising model or $Q = 2$.

$$\sigma\sigma \simeq 1 + \phi_\epsilon \tag{26}$$

So we will obtain

$$\begin{aligned} \langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle &\simeq \langle 11\rangle + \langle\phi_\epsilon\phi_\epsilon\rangle \\ &\simeq 1 + z + \cdots \\ &= P_0 + P_1 + P_2 + P_3 \end{aligned} \tag{27}$$

So the spin terms from P_0 and P_1 cancel each other !

- $Q = 3$

$$\sigma\sigma \simeq 1 + \sigma + \phi_\epsilon + \phi_X + \phi_Y + \phi_Z \tag{28}$$

and then

$$\begin{aligned} \langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle &\simeq 1 + \alpha_0/3z^{2\Delta_\sigma} + \cdots \\ &= P_0 + (2/3)(P_1 + P_2 + P_3) \end{aligned} \tag{29}$$

Potts models in two dimensions

- $Q = 1$

$$\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = P_0 \simeq \alpha_0 r^{2\Delta_\sigma} \tag{30}$$

so no more identity !

- Having no identity is puzzling at first. Indeed, in the usual bootstrap, one always consider

$$\sigma\sigma \simeq 1 + \sum_i C^i_{\sigma\sigma}\varphi_i \tag{31}$$

which corresponds to saying $C^0_{\sigma\sigma} = 1$ or equivalently saying that

$$\langle\sigma(z_1)\sigma(z_2)\rangle \simeq |z_1 - z_2|^{-4\Delta} \tag{32}$$

Conformal bootstrap for the Potts models

- Global conformal symmetry:

$$P_\sigma(\{z_i\}) = |z_1 - z_3|^{-4\Delta_\sigma} |z_2 - z_4|^{-4\Delta_\sigma} P_\sigma \left(\underbrace{\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}}_{\equiv z} \right)$$

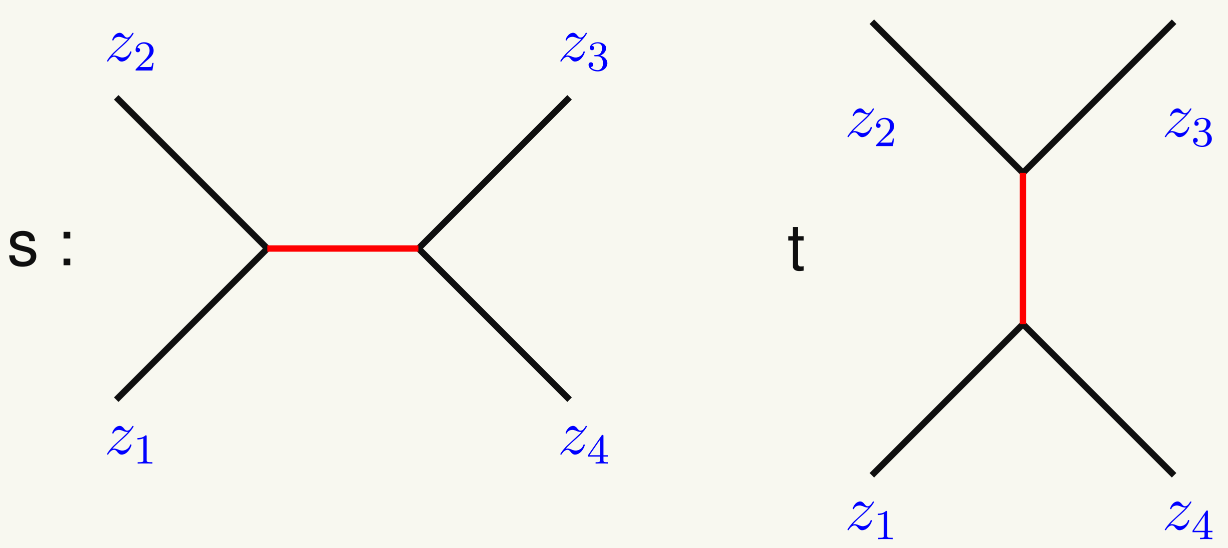
- Symmetry under point permutations

$$P_0(z_1, z_2, z_3, z_4) = P_0(z_1, z_3, z_2, z_4) = P_0(z_1, z_3, z_4, z_2)$$
$$P_1(z_1, z_2, z_3, z_4) = P_2(z_1, z_3, z_2, z_4) = P_3(z_1, z_3, z_4, z_2)$$

Conformal bootstrap for the Potts models

Mixing global conformal invariance and permutation symmetry

channel	limit	permutation	Cross ratio
s	$z_1 \to z_2$	id	z
t	$z_1 \to z_4$	(13)	$1 - z$
u	$z_1 \to z_3$	(14)	$z/(z-1)$



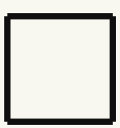
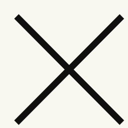
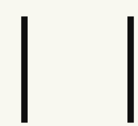
Conformal bootstrap for the Potts models

We have the following relation for the P's :

$$P_0(z) \underbrace{=}_{s-t \text{ symmetry}} P_0(1-z) \underbrace{=}_{s-u \text{ symmetry}} |1-z|^{-4\Delta_\sigma} P_0\left(\frac{z}{z-1}\right)$$

$$P_1(z) = P_3(1-z) = |1-z|^{-4\Delta_\sigma} P_1\left(\frac{z}{z-1}\right)$$

$$P_2(z) = P_2(1-z) = |1-z|^{-4\Delta_\sigma} P_3\left(\frac{z}{z-1}\right)$$

P_0 :  P_1 : P_2 :  P_3 : 

Conformal bootstrap for the Potts models

We start from the variable z :

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \tag{33}$$

and we exchange z_1 with z_3 :

$$\begin{aligned} w &= \frac{(z_3 - z_2)(z_1 - z_4)}{(z_3 - z_1)(z_2 - z_4)} = -\frac{(z_3 - z_2)(z_1 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \\ &= -\frac{(z_1 - z_2)(z_3 - z_2)}{(z_1 - z_3)(z_2 - z_4)} - \frac{(z_2 - z_4)(z_3 - z_2)}{(z_1 - z_3)(z_2 - z_4)} \\ &= -\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} - \frac{(z_1 - z_2)(z_4 - z_2)}{(z_1 - z_3)(z_2 - z_4)} - \frac{(z_2 - z_4)(z_3 - z_2)}{(z_1 - z_3)(z_2 - z_4)} \\ &= -z + \frac{(z_1 - z_2)}{(z_1 - z_3)} - \frac{(z_3 - z_2)}{(z_1 - z_3)} = -z + \frac{(z_1 - z_2)}{(z_1 - z_3)} + \frac{(z_2 - z_3)}{(z_1 - z_3)} \\ &= 1 - z \end{aligned} \tag{34}$$

Conformal bootstrap for the Potts models

We look for functions R_σ that can be related to the P_σ:

$$R_0 \propto P_0 + \mu_s \left(P_1 + P_2 + P_3\right), \quad R_1 \propto P_0 + \mu_{ss} \left(P_2 + P_3\right) + \mu P_1, \ldots$$

The most general form consistent with local conformal symmetry

$$R_\sigma = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} D_{\Delta,\bar{\Delta}} \mathcal{F}_\Delta(\{z_i\}) \mathcal{F}_{\bar{\Delta}}(\{\bar{z}_i\}) \tag{35}$$

$\mathcal{F}_\Delta(\{z_i\})$ Virasoro block with $\Delta_{(0,\frac{1}{2})}$ primary fields and internal field Δ

Conformal bootstrap for the Potts models

Game rules:

Finding the spectrum $\mathcal{S}$ and the structure constants $D_{\Delta,\bar{\Delta}}$ consistent with R_σ.

Ex. $R_2(z) = R_2(1-z)$

$$\sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} D_{\Delta,\bar{\Delta}} \left(\mathcal{F}_\Delta(z)\mathcal{F}_{\bar{\Delta}}(\bar{z}) - \mathcal{F}_\Delta(1-z)\mathcal{F}_{\bar{\Delta}}(1-\bar{z})\right) = 0\ . \tag{36}$$

Conformal bootstrap for the Potts models

Guessing the spectrum $\mathcal{S}^{(k)}$...

- We can show that the single-valuedness of correlation functions imposes the following condition on the spectrum

$$\Delta - \bar{\Delta} \in \frac{1}{2}\mathbb{Z} \tag{37}$$

- An other important result that can be shown : if same spectrum and structure constant in two channels, the spectrum is the same in the third channel if and only if the spectrum is even, *i.e.*

$$\Delta - \bar{\Delta} \in 2\mathbb{Z} \tag{38}$$

Ok for P_0. To match the P_σ, $\sigma = 1, 2, 3$, we have to include also odd spins.

Conformal bootstrap for the Potts models

Our ansatzes

$$\Delta_{(r,s)} = \frac{c-1}{12} + \frac{1}{4}\left(r\beta - \frac{s}{\beta}\right)^2 . \tag{39}$$

$$\mathcal{S}_{X,Y} = \left\{\left(\Delta_{(r,s)}, \Delta_{(r,-s)}\right)\right\}_{r\in X, s\in Y} \qquad \text{with} \qquad \mathrm{X} \subset \mathbb{Z},\ \mathrm{Y} \subset \frac{1}{2}\mathbb{Z} \tag{40}$$

We considered various spectrum based on $\mathcal{S}_{X,Y}$ and found a good agreement with the bootstrap for R_1, R_2, R_3 with the particular case

$$\mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}} \tag{41}$$

This spectrum can also be motivated / justified by the following considerations :

Conformal bootstrap for the Potts models

- A first motivation is that the leading state is the spin operator with the dimension $\Delta_{(0,1/2)} = \Delta_\sigma$
- A second motivation is that such a spectrum was already found for $Q = 4$ which is a special case of the Ashkin-Teller model considered by Al. Zamolodchikov (1986).
- Also, fields with dimensions $\Delta_{(0,\mathbb{Z}+1/2)}$ correspond to the magnetic series identified by Dotsenko & Fateev (1984), Saleur (1987) and Delfino (2013).
- The spectrum $\mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$ also appear in the partition functions computed by Di Francesco, Saleur & Zuber (1987)
- But the main justification, is that it works !!!

Conformal bootstrap for the Potts models

We have found 3 solutions, R_i, $i = 1, 2, 3$ which satisfy the conformal bootstrap equations for all the values of Q.

	s	t	u
R_1	$\mathcal{S}_0$	$\mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$	$\mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$
R_2	$\mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$	$\mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$	$\mathcal{S}_0$
R_3	$\mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$	$\mathcal{S}_0$	$\mathcal{S}_{2\mathbb{Z},\mathbb{Z}+\frac{1}{2}}$

Here, $\mathcal{S}_0$ corresponds to the sector which we have not found yet (but are still looking for ...)

Conformal bootstrap for the Potts models

For instance we found for R_2 and $Q = 1$:

(r, s)	$(\Delta, \bar{\Delta})$	$D_{\Delta,\bar{\Delta}}(24)$	$c_{\Delta,\bar{\Delta}}(24)$
$(0, \frac{1}{2})$	$(\frac{5}{96}, \frac{5}{96})$	1.0000000000	0
$(-2, \frac{1}{2})$	$(\frac{39}{32}, \frac{7}{32})$	0.0385548052	1.3×10^{-8}
$(2, \frac{1}{2})$	$(\frac{7}{32}, \frac{39}{32})$	0.0385548052	1.3×10^{-8}
$(0, \frac{3}{2})$	$(\frac{77}{96}, \frac{77}{96})$	-0.0212806512	4.1×10^{-8}
$(-2, \frac{3}{2})$	$(\frac{95}{32}, -\frac{1}{32})$	0.0004525024	1.2×10^{-7}
$(2, \frac{3}{2})$	$(-\frac{1}{32}, \frac{95}{32})$	0.0004525024	1.2×10^{-7}
$(0, \frac{5}{2})$	$(\frac{221}{96}, \frac{221}{96})$	-0.0000356379	2.5×10^{-6}
$(-4, \frac{1}{2})$	$(\frac{119}{32}, \frac{55}{32})$	-0.0000029746	1.2×10^{-5}
$(4, \frac{1}{2})$	$(\frac{55}{32}, \frac{119}{32})$	-0.0000029746	1.2×10^{-5}

Conformal bootstrap for the Potts models

- This is obtained by solving the equation :

$$\sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} D_{\Delta,\bar{\Delta}} \left(\mathcal{F}_{\Delta}(z)\mathcal{F}_{\bar{\Delta}}(\bar{z}) - \mathcal{F}_{\Delta}(1-z)\mathcal{F}_{\bar{\Delta}}(1-\bar{z})\right) = 0 \,. \quad (42)$$

  with the condition $D_{(0,1/2),(0,1/2)} = 1$ (normalization).

- Truncation at the level N in the number of fields.
- Compute the conformal blocks $\mathcal{F}_{\Delta}(z)$ with Zamolodchikov's recursive formula.
- Select $N-1$ random values of z_i and solve eq.(42) : $D^N_{\Delta,\bar{\Delta}}$
- Repeat the same operation with other set of random values z_i and compute the variance other the different results of $D^N_{\Delta,\bar{\Delta}}$. If these variances, $c_{\Delta,\bar{\Delta}}$ remains small, it is ok.
- Take the limit $\lim_{N\to\infty} D^N_{\Delta,\bar{\Delta}}$. $N=24$ in the previous results for $R=2, Q=1$.

Conformal bootstrap for the Potts models

Comparison with Monte-Carlo calculations : Linear relations between R_σ and P_i

$$R_\sigma = \lambda \left(P_0 + \mu P_\sigma \right) , \quad (\sigma = 1, 2, 3) \tag{43}$$

q	λ	μ
1.	0.9563	-2.0
1.25	0.9426	-3.32
1.5	0.9281	-5.95
1.75	0.9142	-13.85
2.25	0.8881	9.05
2.5	0.8722	4.46
2.75	0.8555	3.48
3.	0.8385	2.0

Conformal bootstrap for the Potts models

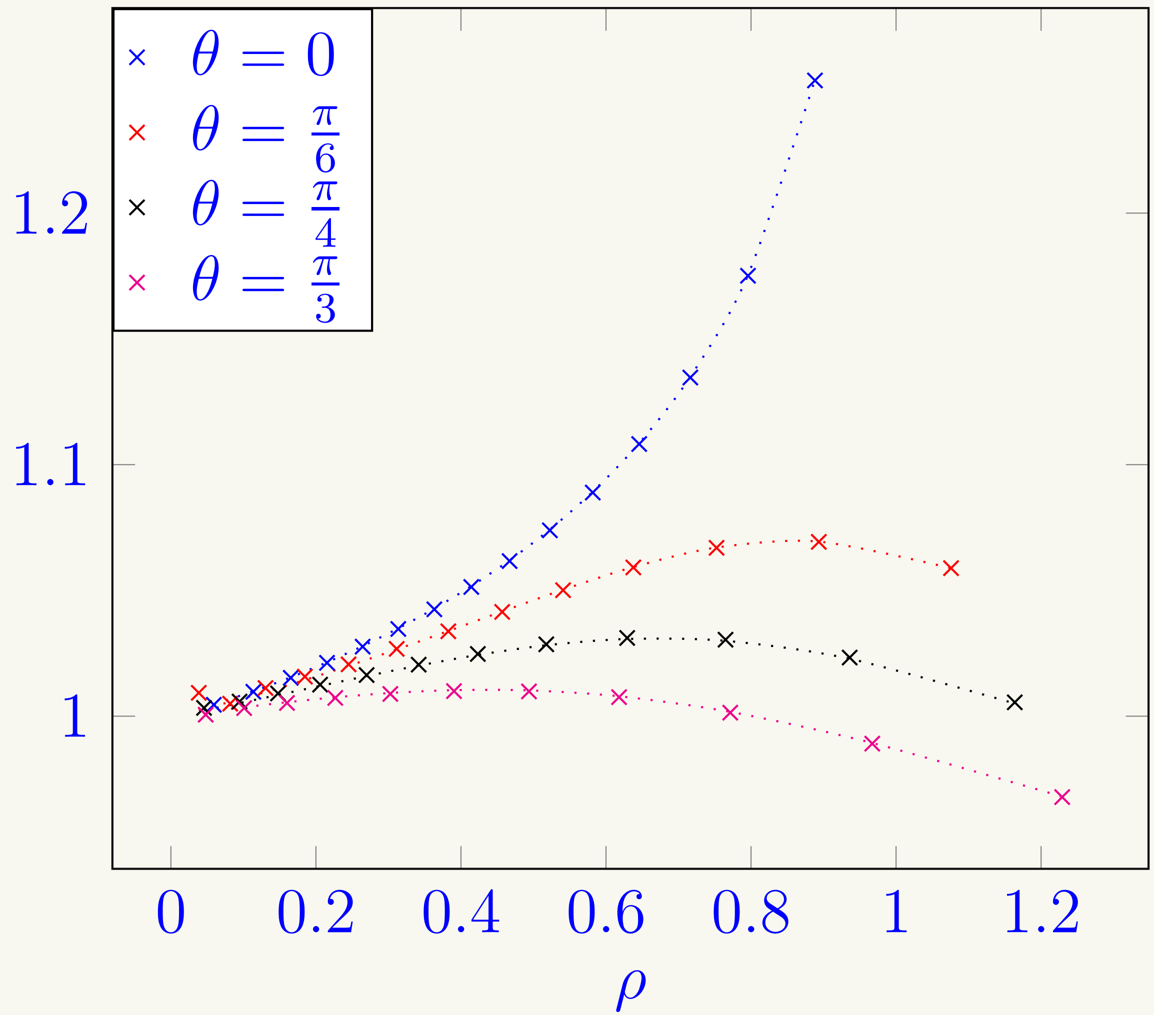
- The parameters λ and μ are obtained by fitting the solution of the bootstrap with combination of the Monte Carlo simulations for the following correlation functions :

$$\rho^{2\Delta_{(0,\frac{1}{2})}} R_2\left(\rho e^{i\theta}, 0, \infty, 1\right) \tag{44}$$

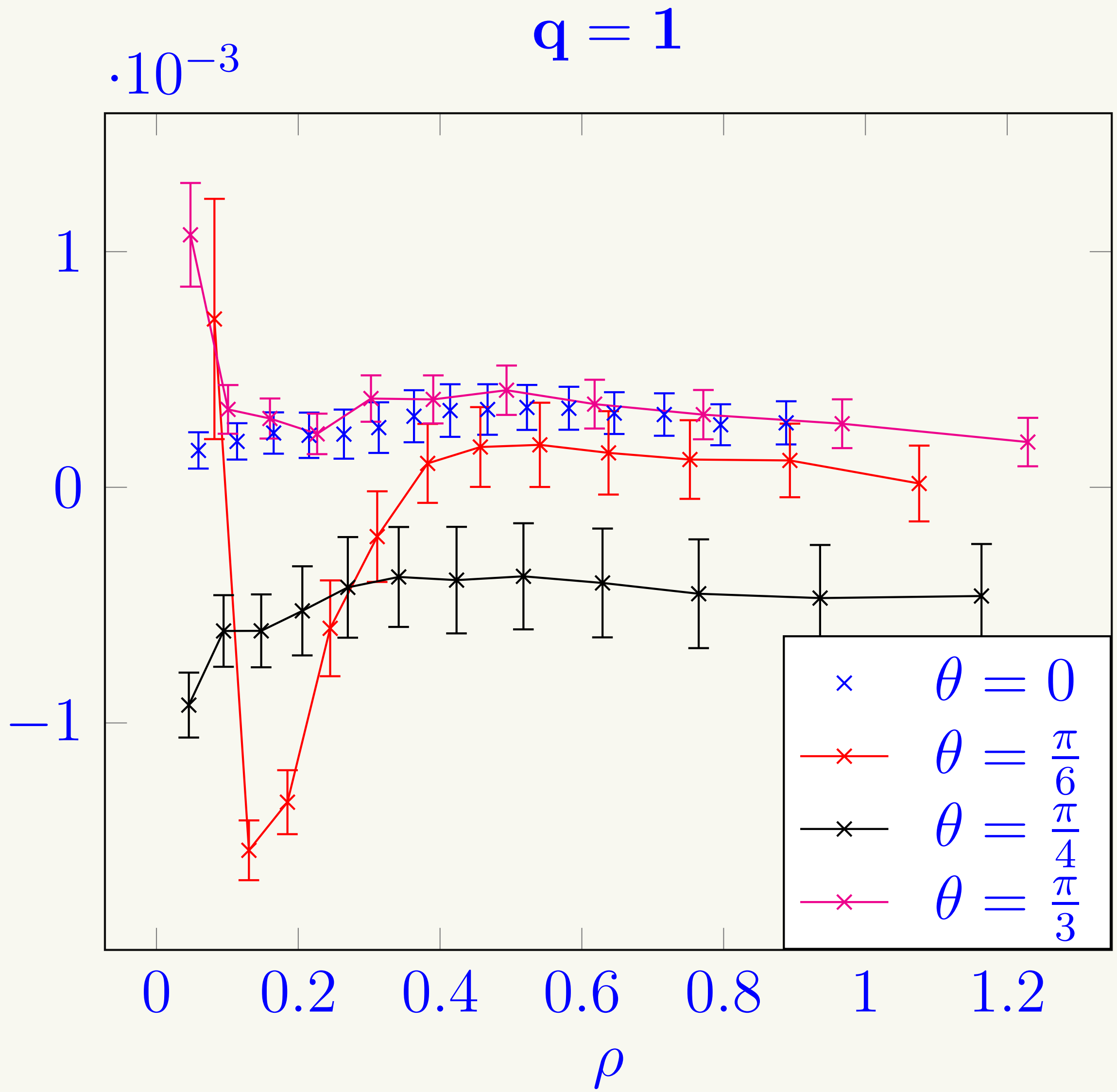
- $Q = 1$ for various values of θ. For that case, λ and μ do not depend on θ.
- $\theta = 0$ for various values of Q

Conformal bootstrap for the Potts models

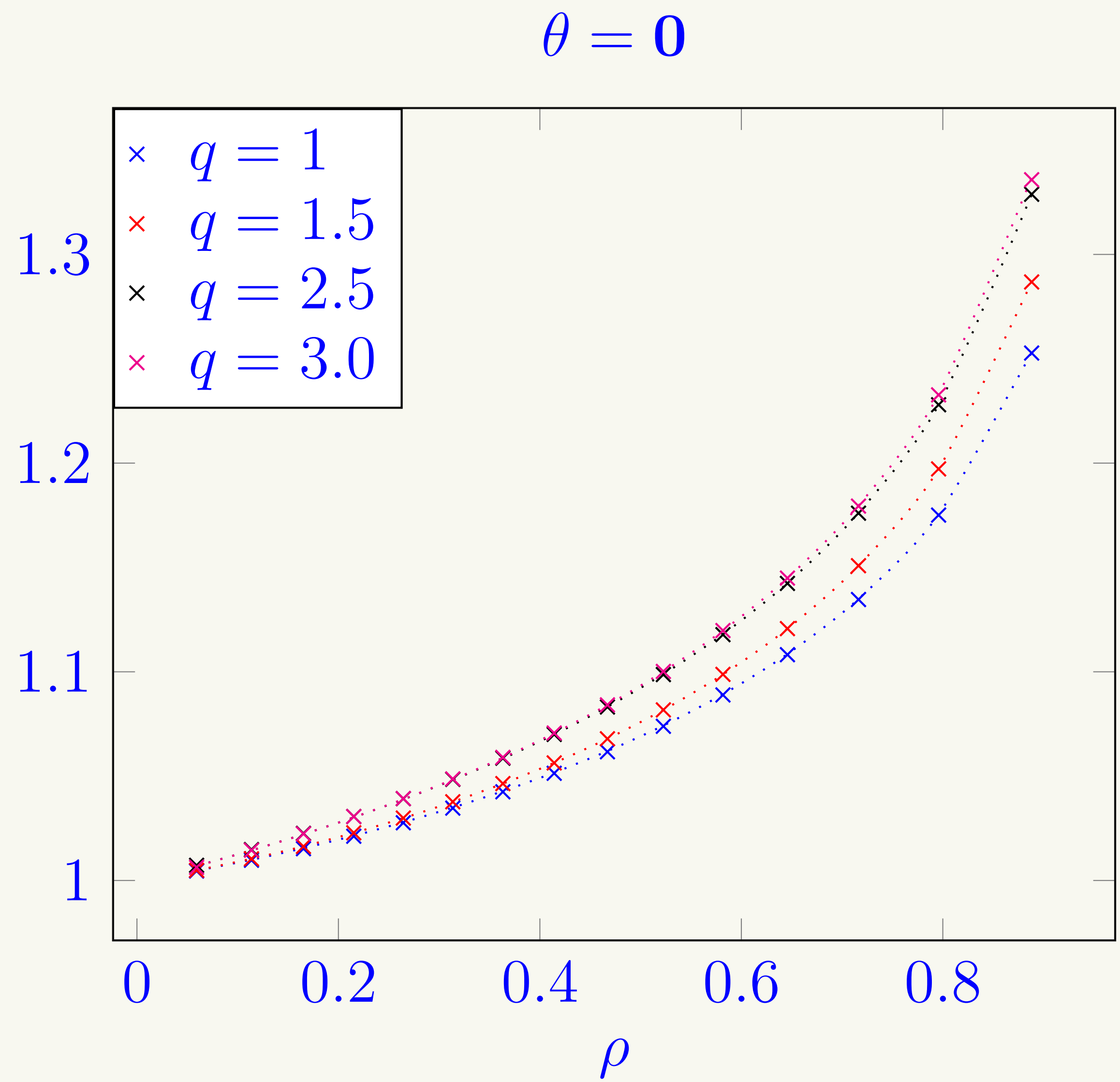
$\mathbf{q = 1}$



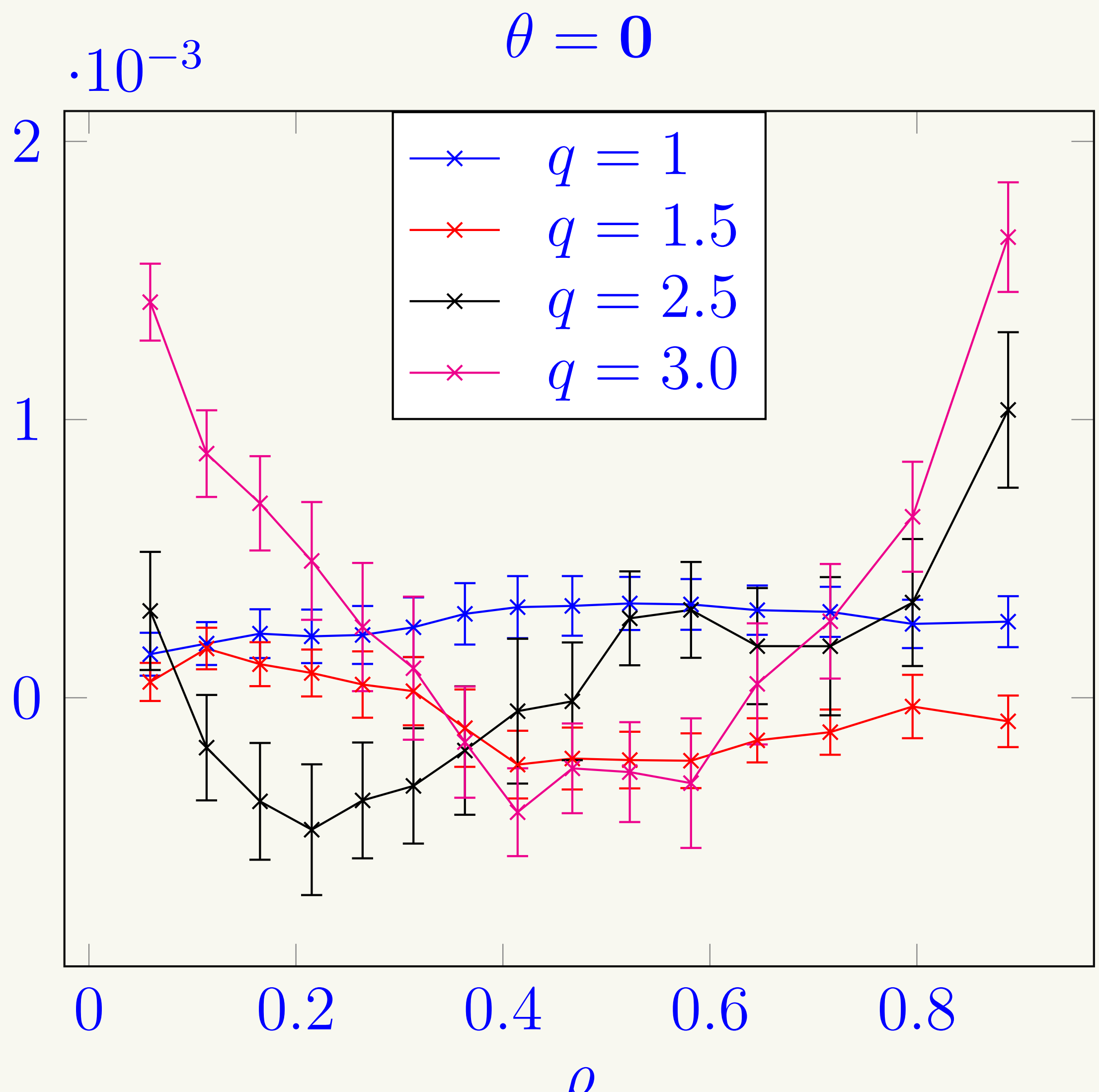
Conformal bootstrap for the Potts models



Conformal bootstrap for the Potts models



Conformal bootstrap for the Potts models



Conformal bootstrap for the Potts models

A series of comments:

- The conformal boostrap solutions match, in the case of $Q = 3$ with W_3 correlation function, and at $Q = 4$ with Zamolodchikov solutions for Ashkin-Teller model.
- For $Q = 4$, Zamolodchikov also found the fourth solution *i.e.*

$$\mathcal{S}_0 = \mathcal{S}_{2\mathbb{Z},\mathbb{Z}} \tag{45}$$

  But this does not work in general. We are still looking for such a solution for general Q !!!
- On general grounds, we can expect that the ground state is the identity (*i.e.* $\Delta = 0$) in $\mathcal{S}_0$. This is indeed the case for Zamolodchikov solution for $Q = 4$. For other values of Q, $\mathcal{S}_{2\mathbb{Z},\mathbb{Z}}$ would contain operators with negative dimension.

Conclusion

- 2D Conformal bootstrap approach provided new four point functions that are in excellent agreement with Monte Carlo results. No logarithmic features so far.

- A continent to explore: i) determine $\mathcal{S}_0$, ii) Liouville $c \leq 1$ ($C^{c\leq 1}$) play a role? iii) other probabilities,,,

- Available codes at https://github.com/ribault/bootstrap-2d-Python