Entanglement in CFTs

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1. Entanglement in Quantum Mechanic
   - Definition and Properties
   - Entanglement Measures
2. Entanglement in QFT
   - Geometric (Entanglement) Entropy
   - Main Properties of EE
3. Entanglement in CFTs
   - Entanglement Measures in 2d CFT
   - Quantum Quench and Time Evolution of EE

Motivation

Entanglement is important in condensed matter, quantum information and black hole physics.

- Quantum phase transitions at $T = 0$
- Non-equilibrium processes in strongly coupled field theories
- Improve the connection between gauge theory and gravity
- $\cdots$

Entanglement in Quantum Mechanic

(Pure) Entangled states

- Consider two quantum systems A and B

$$\mathcal{H}_A, \quad |i\rangle, \quad i = 1, \cdots, n, \quad \mathcal{H}_B, \quad |a\rangle, \quad a = 1, \cdots, m$$

- Construct M using the tensor product of A and B

$$\mathcal{H}_M = \mathcal{H}_A \otimes \mathcal{H}_B, \quad |i\rangle \otimes |a\rangle \equiv |i, a\rangle$$

- (Pure) Entangled states

$\exists \; |\chi\rangle \in \mathcal{H}_M :$

$$|\chi\rangle_{\mathcal{H}_M} \neq |\psi\rangle_{\mathcal{H}_A} \otimes |\phi\rangle_{\mathcal{H}_B}$$

(Mixed) Entangled states

- Consider M in a mixed state described by a density matrix

$$\rho_M = \sum_i c_i \big|\chi_i\rangle\langle\chi_i\big|$$

- Definition of reduced density matrix for A

$$\rho_A \equiv \mathrm{Tr}_B(\rho_M) = \sum_{i=1}^{dim[B]} \langle\phi_i\big|\rho_M\big|\phi_i\rangle$$

- (Mixed) Entangled states

$$\rho_M \neq \sum_i p_i \, \rho_A^{(i)} \otimes \rho_B^{(i)}$$

Entanglement Measures

1. Entanglement entropy (EE): von-Neumann entropy for ρ_A

$$S_A = -\mathrm{Tr}_A \left(\rho_A \log \rho_A\right)$$

- Pure state $\qquad S(A) = S(\bar{A})$
- Subadditivity $\qquad S(A) + S(B) \geq S(A \cup B)$

2. Mutual information (MI)

$$I(A_1, A_2) = S_{A_1} + S_{A_2} + S_{A_1 \cup A_2}$$

3. Tripartite information, Relative entropy, Negativity, $\cdots$

Entanglement Entropy (EE)

- Example: spin 1/2 particle

$$\begin{cases} |\Psi_1\rangle = |\uparrow\rangle_A \otimes |\downarrow\rangle_B \\ |\Psi_2\rangle = |\downarrow\rangle_A \otimes |\uparrow\rangle_B \end{cases}, \qquad S_A = 0$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\Psi_1\rangle + |\Psi_2\rangle\right), \qquad S_A = \log 2$$

EE is a state dependent and non-linear operator in QM.

Challenge

Generalization to QFT (Continuum Limit)?

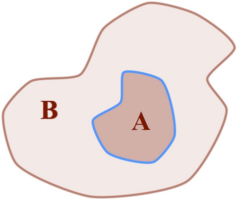
Entanglement in QFT

Different Notions of EE in QFT

- Different Hilbert space decompositions lead to different types of EE
  1. Geometric (Entanglement) entropy
     [Bombelli-Koul-Lee-Sorkin '86, Srednicki '93, Callan-Wilczek '94]
  2. Momentum space entanglement entropy
     [Balasubramanian-McDermott-Van Raamsdonk '11]
  3. Field space entanglement entropy
     [Yamazaki '13]

Geometric entropy

- Consider a local d-dimensional QFT on $\mathbb{R} \times \mathcal{M}^{(d-1)}$
- Divide $\mathcal{M}^{(d-1)}$ into two parts

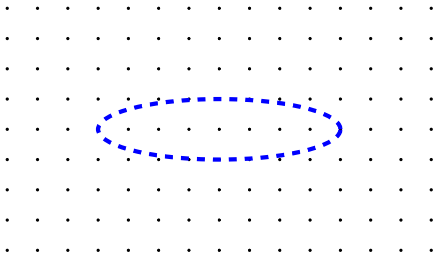


- Locality implies $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$S_A = -\mathrm{Tr}_A \left(\rho_A \log \rho_A\right)$$

Main Properties of EE

- Extensive limit (Thermal entropy) $A \to$ Total system
- Area law $S_A \propto \frac{\text{Area}(\partial A)}{\epsilon^{d-2}}, \quad (d > 2)$



Main Properties of EE

- Extensive limit (Thermal entropy) $\qquad A \to$ Total system
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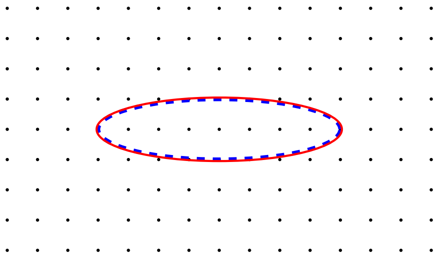
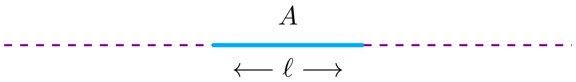


Geometric entropy

- Area law violation

1. Logarithmic divergence in 2-dim CFT [Calabrese-Cardy, '04]



2. Logarithmic divergence for fermions [Wolf '06, Gioev-Klich '06]
3. Volume law for non-local QFTs [Shiba-Takayanagi '13]

Renyi Entropy and Replica Trick

- $S_A = -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right)$

  Taking the logarithm of ρ_A is very complicated!

- Renyi Entropy

$$S_A^{(n)} = \frac{1}{1-n}\mathrm{Tr}_A\left(\rho_A^n\right)$$

- Replica Trick

$$S_A = \lim_{n\to 1} S_A^{(n)}$$

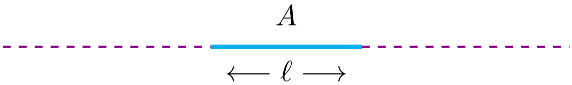
[Bombelli-Koul-Lee-Sorkin '86, Callan-Wilczek '94]

Challenge

Conformal symmetry may help us to more simplify the problem!

Entanglement in CFTs

EE in $2d$ CFT



EE for Pure State

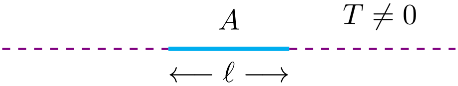
$$S_A = \lim_{n\to 1} \frac{1}{1-n} \log\langle \Phi_n(0)\Phi_{-n}(\ell)\rangle$$

- $S_A = \frac{c}{3} \log \frac{\ell}{\epsilon}$,

1. Only depends on c
2. A divergent quantity

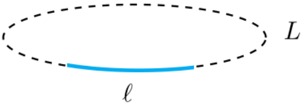
[Calabrese-Cardy, '04]

EE in $2d$ CFT



1. EE for a Thermal State

$$S_A = \frac{c}{3} \log \left(\frac{\beta}{\pi \epsilon} \sinh \frac{\pi \ell}{\beta} \right)$$



2. EE for a Finite System $(x \sim x + L)$

$$S_A = \frac{c}{3} \log \left(\frac{L}{\pi \epsilon} \sin \frac{\pi \ell}{L} \right)$$

[Calabrese-Cardy, ’04]

MI in $2d$ CFT

Mutual information

$$I(A,B) = S_A + S_B - S_{A\cup B}$$

$$S_{A\cup B} = \lim_{n\to 1} \frac{1}{1-n} \log\langle \Phi_n \Phi_{-n} \Phi_n \Phi_{-n} \rangle$$

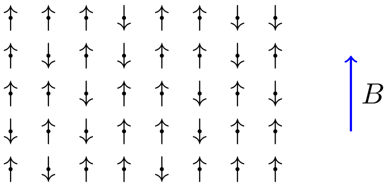
1. Depends on full operator content
2. A finite quantity
3. Positive definite

[Calabrese-Cardy-Tonni, '09]

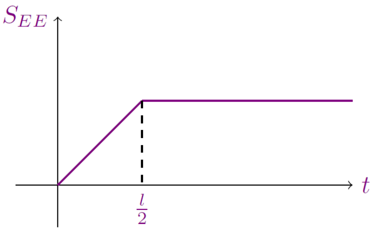
Quantum Quench

Quantum Quench

$$\begin{array}{ccc} t<0 & t=0 & t>0 \\ H(\lambda_0) & \longrightarrow & H(\lambda) \\ |\psi_0\rangle & \longrightarrow & e^{-iH(\lambda)t}|\psi_0\rangle \end{array}$$



Time Evolution of EE in CFT_2



- Description in terms of free streaming particles
  [Calabrese-Cardy, '05]

Further Studies

Challenges

1. Extension to higher dimensions and general entangling regions (Heat Kernel method only works for free theories!)
2. Investigating the role of symmetry
   - Lifshitz and hyperscaling-violating exponents, ...
3. Considering other entanglement measures
   - Relative entropy, Renyi entropy, ...

Holography may help us to overcome these problems

Thanks