Aspects of Fractional CFTs

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Overview

Motivation

In this talk I will try to show how the Fractionality comes to play in the thermodynamical properties as well as the quantum characteristics of a system.

Doing so we investigate:

- Field Theories on the Fractals
- A toy model of a Fractional field theory

Based on "A.F.A, PRD. 16" and a work in progress.

EE in QM

- Consider a quantum mechanical system in a pure ground state which is described by $|\psi\rangle$ ($\rho = |\psi\rangle\langle\psi|$).

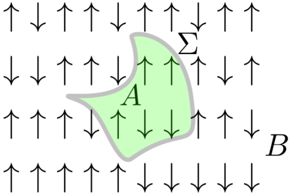


Figure : Note: Σ is imaginary!

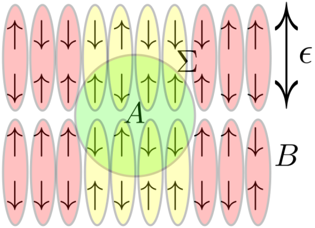
- *Reduced* density operator:

$$\rho_A = \mathrm{Tr}_B\, \rho = \mathrm{Tr}_B |\psi\rangle\langle\psi|\,.$$

Then the EE is

$$S_{EE}(A) = -\mathrm{Tr}\rho_A \log \rho_A\,.$$

EE in QFT



- Area law is an interesting feature [Srednicki (03)]

$$S_{EE} \sim \frac{\mathcal{A}(\Sigma)}{\epsilon^{d-2}}\,.$$

EE in general d

$$S_{EE}(\Sigma) = \frac{s_{d-2}}{\epsilon^{d-2}} + \frac{s_{d-4}}{\epsilon^{d-4}} + \cdots + s_0 \log \epsilon + f\ ,$$

generally

$$s_i = s_i(\mathcal{R}, \mathcal{K})$$

in particular

$$s_{d-2} \propto \mathcal{A}(\Sigma).$$

Note: In two dimensions $s_0 = -\frac{c}{3}$.

Rényi entropy

In a QFT, we first construct the Rényi entropy as

$$S_{RE}(A) = \frac{1}{1-n} \log \mathrm{Tr} {\rho_A}^n \,,$$

The EE reads then

$$S_{EE}(A) = \lim_{n \to 1} S_{RE}(A).$$

Partition function on $\mathcal{R}_n$

Finding the RE reduces to computing the partition function on n-sheeted Riemann surface

$$\mathrm{Tr}\hat{\rho}_A^n = Z_1^{-n} \int_{\mathcal{R}_n} \mathcal{D}\phi \, e^{-\int_{\mathcal{R}_n} d\tau \mathcal{L}[\phi]} \equiv \frac{Z_n}{Z_1^n} \, ,$$

then after an analytical continuation in n we will have

$$S_{EE}(A) = -\,\mathrm{Tr}\hat{\rho}_A \log \hat{\rho}_A = -\partial_n \log \, \mathrm{Tr}\rho_A^n\big|_{n=1} = -(n\partial_n - 1) \log Z_n\big|_{n=1} \, .$$

But the deficit angle $\alpha = 2\pi(1-n)$ introduces a conical singularity such that $\mathcal{R}_n \sim \mathcal{C}_n \times \Sigma$.

• The main challenge: **calculation on a manifold with conical singularity.**

Heat Kernel

$$K(x, x'; s) = \langle x | e^{-s\Delta} | x' \rangle \, .$$

$$(\partial_s + \Delta) K(x, x'; s) = 0 \, ,$$

with initial condition

$$K(x, x'; 0) = \delta(x - x') \, .$$

Then

$$W = -\log Z = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \, \mathrm{Tr} K \, .$$

Thermodynamics:

$$\log Z_\beta = \frac{1}{2}\int_0^\infty \frac{ds}{s}\left(\mathrm{Tr}K_{S^1} \times \mathrm{Tr}K_{\mathbb{R}^{d-1}}\right),$$

Entanglement Pattern:

$$W_n = -\log Z_n = -\frac{1}{2}\int_{\epsilon^2}^\infty \frac{ds}{s}\left(\mathrm{Tr}K_{\mathcal{C}^2} \times \mathrm{Tr}K_{\mathbb{R}^{d-2}}\right),$$

Sommerfeld Formula:

$$\mathrm{Tr}K_{\mathcal{C}^2} = \mathrm{Tr}K(s)_{\mathbb{R}^2} + \frac{i}{4\pi n}\int_\Gamma d\omega \int d^2x \cot\left(\frac{\omega}{2n}\right) K_{\mathbb{R}^2}(r' = r, \tau' = \tau + \omega)\,.$$

EE for a scalar theory in d dimensions:

$$S_{EE} = \frac{1}{6(d-2)}\frac{V_{d-2}}{\epsilon^{d-2}}\,.$$

Fractals $\mathscr{F}_{d_w}^{d_f}$

A fractal is a *self similar* object with a characteristic dimension known as the *fractal dimension* which exceeds its topological dimension

- *The Fractal (Hausdorff) dimension:*

$$d_f = \lim_{\ell \to 0} \frac{\log V(\ell)}{\log \ell} \,,$$

- *The Walk dimension:*

Walk dimension is the index of the anomalous diffusion on the fractal in the sense that

$$x_{r.m.s} \propto t^{1/d_w} \,.$$

in this sense $\mathbb{R}^d$ is just a particular fractal, $\mathbb{R}^d \sim \mathscr{F}_2^d$.

Entropic Fractals

$$K_{\mathscr{F}^{d_f}_{d_w}}(X, X'; s) \approx \frac{1}{(4\pi s)^{d_f/d_w}} \exp\left[-\left(\frac{|X - X'|^{d_w}}{4s}\right)^{\frac{1}{d_w-1}}\right],$$

Entanglement Entropy:

$$S_{EE}(\mathscr{F}^{d_s}) = \frac{1}{6}\frac{1}{(4\pi)^{d_s/2}}\frac{A_s(\Sigma)}{d_s \epsilon^{d_s}}.$$

Thermal Entropy:

$$S_T(\mathscr{F}^{d_s}) = \frac{1}{2^{d_s-1}\pi^{(3d_s+1)/2} R^{d_s}} \Gamma\left(\frac{d_s+3}{2}\right) \zeta_R(d_s+1) V_s\,,$$

Important Features

Entanglement Entropy:

$$S_{EE}(\mathscr{F}^{d_s}) \sim \frac{1}{d_s \epsilon^{d_s}} \,.$$

Thermal Entropy:

$$S_T(\mathscr{F}^{d_s}) \sim T^{d_s} \,.$$

Novel log behavior

$$S_{EE}(\mathscr{F}) \to S_{EE}(\mathscr{F})(1 + a\cos(b\log\epsilon + c)) + \cdots .$$

Fractality$\sim$Complex dimension.

Holographic EE

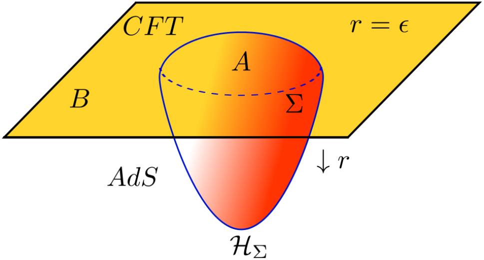


Figure : Ryu-Takayanagi's (RT) proposal (06)

$$S_{HE}(\Sigma) = \text{Min}\, \frac{\mathcal{A}(\mathcal{H}_\Sigma)}{4G_N^{(d+1)}} \,,$$

Hyperscaling Violating Geometries

$$ds^2 = r^{-\frac{-2(d-\theta+1)}{(d+1)}}(-dt^2 + dr^2 + \sum_{i=1}^{d-1} dx_i^2)\,,$$

Entanglement Entropy:

$$S_{EE} \sim \frac{1}{d_\theta \epsilon^{d_\theta}}\,.$$

Thermal Entropy:

$$S_T \sim T^{d_\theta}\,.$$

So I propose that

$$\boxed{d_s \sim d_\theta\,.}$$

A toy model of a Fractional CFT

Consider a field theory with the fractional Laplacian

$$S = \frac{1}{2} \int d^d x \phi(x) \pounds^{\frac{\alpha}{2}} \phi(x) \, ,$$

This is a well known example of non-local field theories which generally admits the conformal symmetry for $\alpha < d$.

W

e cane define the fractional Laplacian through the Fourier transformation

$$\pounds^{\frac{\alpha}{2}} \phi(p) = |p|^{\alpha} \phi(p) \, .$$

Fortunately, there is a good estimate for the heat kernel of the fractional Laplacian which reads

$$K(x,x';s) = \frac{c}{s^{\frac{d}{\alpha}}}\left(1+\frac{|x-x'|^{\alpha}}{s}\right)^{-\frac{d+\alpha}{\alpha}} \quad .$$

Remember that Thermodynamics:

$$\log Z_\beta = \frac{1}{2}\int_0^\infty \frac{ds}{s}(\,\mathrm{Tr}K_{S^1}\times\,\mathrm{Tr}K_{\mathbb{R}^{d-1}})\,,$$

Entanglement Pattern:

$$W_n = -\log Z_n = -\frac{1}{2}\int_{\epsilon^2}^\infty \frac{ds}{s}(\,\mathrm{Tr}K_{\mathcal{C}^2}\times\,\mathrm{Tr}K_{\mathbb{R}^{d-2}})\,,$$

Thermodynamics

$$U = -\frac{\partial}{\partial\beta}\log Z_\beta = (d-1)c^2 V_{d-1}^{\frac{2}{\alpha}} \frac{\Gamma\left(\frac{d}{\alpha}\right)\Gamma\left(\frac{\alpha-d+1}{\alpha}\right)}{\Gamma\left(\frac{\alpha+1}{\alpha}\right)}\zeta(d)\beta^{-d}\,,$$

$$P = \frac{1}{\beta}\left(\frac{\partial}{\partial V_{d-1}}\log Z_\beta\right)_\beta = \frac{2}{\alpha}c^2 V_{d-1}^{\frac{2-\alpha}{\alpha}} \frac{\Gamma\left(\frac{d}{\alpha}\right)\Gamma\left(\frac{\alpha-d+1}{\alpha}\right)}{\Gamma\left(\frac{\alpha+1}{\alpha}\right)}\zeta(d)\beta^{-d}\,.$$

Interestingly, these function satisfy the usual equation of state, i.e.

$$PV_{d-1} = \frac{2}{\alpha}\frac{1}{d-1}U\,.$$

Thermal entropy:

$$S_{Th} = -(\beta\partial_\beta - 1)\log Z_\beta = dc^2 V_{d-1}^{\frac{2}{\alpha}} \frac{\Gamma\left(\frac{d}{\alpha}\right)\Gamma\left(\frac{\alpha-d+1}{\alpha}\right)}{\Gamma\left(\frac{\alpha+1}{\alpha}\right)}\zeta(d)\beta^{1-d}\,.$$

Entanglement pattern

$$S_{EE} = \frac{\pi c^2}{12} \frac{\alpha}{d-2} \frac{V_{d-2}^{\frac{2}{\alpha}}}{\epsilon^{\frac{2(d-2)}{\alpha}}} \,.$$

In the case of $\alpha = 2$, i.e. for the usual Laplacian we get

$$S_{EE}(\alpha = 2) = \frac{\pi c^2}{6} \frac{1}{d-2} \frac{V_{d-2}}{\epsilon^{d-2}} \,.$$

central charge

$$C_f = \frac{\alpha}{2} C \ ?$$

Thanks!