Conformal Bootstrap

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- Model to describe critical phenomena(phase transition in ferro-magnetism).
- Based on a spin lattice with nearest-neighbours interactions:

$$H = -J \sum_{i} \sum_{i \sim j} \sigma_i \sigma_j$$

.

- Continuum limit: iteratively sum the spins in a block of size L and replace σ_i with the average value.
- At the critical temperature the correlation length diverges, and the sin-spin correlator exhibits scale invariance.

- The emergence of conformal symmetry at the critical point is more mysterious. This semms to be a generic future of criticality but why this happens in not fully.

We will take conformal invariance of the critical point for granted.

- While the 2D Ising model was solved exactly on the lattice for any temperature by Onsager and Kaufman in the 1940's, the 3D lattice case has resisted all attemps for an exact solution.
- Istrail proved in 2000 that solving 3D Ising model on the lattice is an NP-complete problem.
- However, above theorem does not exclude the possibility of finding a solution in the continuum limit.
- The standard way to think about the continuum theory is in terms of local operators (or fields).

- At $T = T_c$, the theory has scale invariance, and each operator is characterized by its scaling dimension Δ and $O(3)$ spin.
- Few notable local operators, which split into odd and even sectors under the global Z_2 symmetry (the Ising spin flip).

Operator	Spin l	$\mathbb{Z}_2$	Δ	Exponent
σ	0	$-$	0.5182(3)	$\Delta = 1/2 + \eta/2$
σ'	0	$-$	$\gtrsim 4.5$	$\Delta = 3 + \omega_A$
ε	0	$+$	1.413(1)	$\Delta = 3 - 1/\nu$
ε'	0	$+$	3.84(4)	$\Delta = 3 + \omega$
ε''	0	$+$	4.67(11)	$\Delta = 3 + \omega_2$
$T_{\mu\nu}$	2	$+$	3	n/a
$C_{\mu\nu\kappa\lambda}$	4	$+$	5.0208(12)	$\Delta = 3 + \omega_{\text{NR}}$

- The operators σ and ϵ are the lowest dimension Z_2-odd and even scalars respectively. These are the continuum space versions of the Ising spin and of the product of two neighbouring spins on the lattice.

- The approximate values of operator dimensions given in the table have been determined from a variety of theoretical techniques, most notably the *epsilon*-expansion,high temperature expansion, and Monte-Carlo simulations.

- Can we do better?

Using just symmetries together with some fundamental assumptions

- Basis of local operators $\mathcal{O}_i$ with scaling dimensions Δ_i.
- $P_\mu = \partial_\mu$

$$\mathcal{O}_\Delta \rightarrow^P \mathcal{O}_{\Delta+1} \rightarrow^P \mathcal{O}_{\Delta+2} \rightarrow^P +...$$

derivative operators (Descendants)

- Special conformal transformation generator $K_\mu = 2x_\mu(x.\partial) - x^2\partial_\mu$

$$\mathcal{O}_\Delta \leftarrow^K \mathcal{O}_{\Delta+1} \leftarrow^k \mathcal{O}_{\Delta+2} \leftarrow^k ...$$

- So each multiplet must contain the lowest-dimension operator:

$$K_\mu.\mathcal{O}_\Delta(0) = 0.$$

Primary operators

- In unitary theories dimensions have lower bounds:

$$\Delta \geq D - 2 + l \quad (\geq D/2 - 1 \ for \ \ l = 0).$$

- Two point functions: fixed completely modulo a normalization constant

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{a}{x_{12}^{2\Delta}}.$$

- Three point function: fixed modulo a constant

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}'(x_3)\rangle = \frac{\lambda_{\Delta,0}}{x_{12}^{2\Delta_\sigma-\Delta}x_{23}^{\Delta}x_{13}^{\Delta}}, \quad [\mathcal{O}=\Delta_\sigma,\ \mathcal{O}'=\Delta]$$

- Four point function: fixed modulo a general function

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = x_{12}^{-2\Delta_\sigma}x_{34}^{-2\Delta_\sigma}g(u,v),$$

where

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, \quad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}.$$

- OPE:

$$\mathcal{O}(x)_{\Delta_1,0}\mathcal{O}(0)_{\Delta_2,0} = \frac{1}{|x|^{\Delta_1+\Delta_2}} \sum_{\Delta,l} \lambda_{\Delta,l} \left(\mathcal{O}'(0)_{\Delta,l} + descendants\right)$$

Use OPE to reduce higher point functions to smaller ones.

- Then

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \sim u^{-\Delta_\sigma} \sum_{\mathcal{O}'_{\Delta,l}} \lambda^2_{\Delta,l} \left(\langle \mathcal{O}'_{\Delta,l}\mathcal{O}'_{\Delta,l}\rangle + descendants\right)$$

Note that unitarity impose that $\lambda^2_{\Delta,l} \geq 0$.

- Conformal Blocks

$$g_{\Delta,l}(u,v) = \langle \mathcal{O}'_{\Delta,l}\mathcal{O}'_{\Delta,l}\rangle + descendants$$

They sum up the contribution of an entire representation.

Old idea (70s) but none could use them for long time, until...

- 2003 $\rightarrow$ Explicit expression for even dimensions
- 2011 $\rightarrow$ Explicit expression for any dimensions,
- For example

$$g_{\Delta,l}(u,v) \sim k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z}), \qquad (D=2)$$

$$g_{\Delta,l}(u,v) \sim \frac{z\bar{z}}{z-\bar{z}}\left[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right], \qquad (D=4)$$

where

$$k_\beta(x) \equiv x^{\beta/2}F_1^2(\beta/2,\beta/2,\beta;x), \qquad u = z\bar{z}, \quad v=(1-z)(1-\bar{z}).$$

- The expressions for odd dimensions are more complicated!.

- Which expansion is the right one?

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \quad vs \quad \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$$

- They must produce the same result. The constraint

$$u^{-\Delta_\sigma}\left(1+\sum_{\Delta,l}\lambda^2_{\Delta,l}\,g_{\Delta,l}(u,v)\right) = v^{-\Delta_\sigma}\left(1+\sum_{\Delta,l}\lambda^2_{\Delta,l}\,g_{\Delta,l}(v,u)\right).$$

- Crossing symmetry $\rightarrow$ Sum Rule

$$\sum_{\Delta,l}\lambda^2_{\Delta,l}\,F_{d,\Delta,l} = 1,$$

where

$$F_{\Delta_\sigma,\Delta,l} = \frac{v^{\Delta_\sigma}g_{\Delta,l}(u,v) - u^{\Delta_\sigma}g_{\Delta,l}(v,u)}{u^{\Delta_\sigma} - v^{\Delta_\sigma}}.$$

- $F_{d,\Delta,l}$ are known functions. $\lambda^2_{\Delta,l}$ are unknown coefficients.

- One can see the above problem is equal to this linear programming

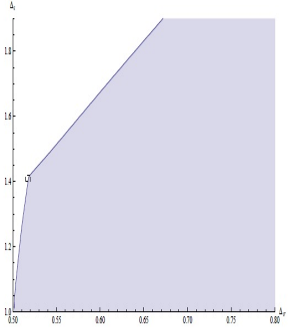
$$min(\tilde{C} = \sum_{i \epsilon S} \tilde{c}_i a_i) \quad s.t. \quad \sum_{i \epsilon S} a_i \mathbf{v}_i = \mathbf{t}, \qquad a_i > 0.$$

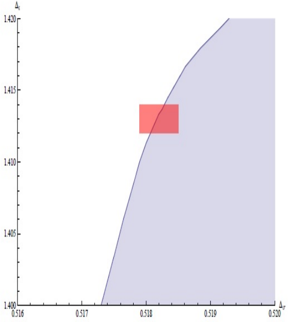
which can be solved using the "primal simplex algorithm".

- Crossing symmetry in

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle$$

- Allowed regions in $\Delta_\epsilon - \Delta_\sigma$ plane :

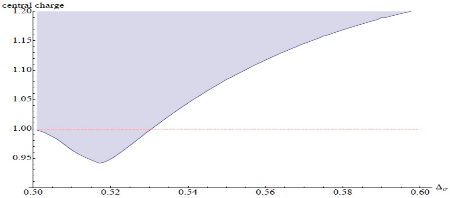




[El-Showk,Paulos,Poland,Rychkov,Simmons-Duffin,Vichi ,2014]

- Allowed values of c_T as function of Δ_σ:

$$c = \frac{D}{D-1}\frac{\Delta_\sigma^2}{\lambda_{d,2}^2},$$



- The minimum is in correspondence of Ising. It predicts $\frac{c_T^{Ising}}{c_T^{free}} \sim 0.94 - 0.95$
- No accurate measurement nor calculation to compare with. ϵ-expansion at first order gives $\frac{c_T^{Ising}}{c_T^{free}} \sim 0.98$

David Polanda,David Simmons-Duffin, 2014

- Combine the crossing equation for $\langle\sigma\sigma\epsilon\epsilon\rangle$ with crossing equations for $\langle\sigma\sigma\sigma\sigma\rangle$ and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$

$$\sum_{\mathcal{O}^+} \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}} & \lambda_{\epsilon\epsilon\mathcal{O}} \end{pmatrix} V_{+,\Delta,l} \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}} \\ \lambda_{\epsilon\epsilon\mathcal{O}} \end{pmatrix} + \sum_{\mathcal{O}^-} \lambda^2_{\sigma\epsilon\mathcal{O}} V_{-,\Delta,l} = 0.$$

$$V_{-,\Delta,l} = \begin{pmatrix} 0 \\ 0 \\ F^{\sigma\epsilon,\sigma\epsilon}_{-,\Delta,l} \\ (-l)^l F^{\epsilon\sigma,\sigma\epsilon}_{-,\Delta,l} \\ -(-l)^l F^{\epsilon\sigma,\sigma\epsilon}_{+,\Delta,l} \end{pmatrix} \qquad V_{+,\Delta,l} = \begin{pmatrix} \begin{pmatrix} F^{\sigma\sigma,\sigma\sigma}_{-,\Delta,l} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & F^{\epsilon\epsilon,\epsilon\epsilon}_{-,\Delta,l} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & F^{\sigma\sigma,\epsilon\epsilon}_{-,\Delta,l} \\ F^{\sigma\sigma,\epsilon\epsilon}_{-,\Delta,l} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & F^{\sigma\sigma,\epsilon\epsilon}_{+,\Delta,l} \\ F^{\sigma\sigma,\epsilon\epsilon}_{+,\Delta,l} & 0 \end{pmatrix} \end{pmatrix}$$

- The above problem is equal to a more standard semidefinite program of the following form:

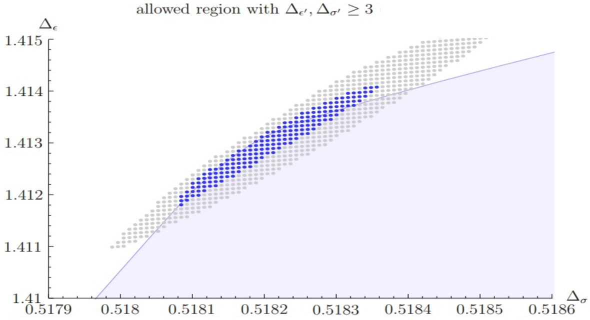
$$\text{maximize} \quad Tr(CY) + b.y \quad \text{over} \quad y \in R^N, Y \in \mathbf{S}^K,$$

$$\text{such that} \qquad Tr(AY) + By = c, \qquad and \qquad Y \succeq 0,$$

where

$$c \in R^P, \quad B \in R^{p\times N}, \quad A_i, C \in \mathbf{S}^K.$$

- One can solve this type of programs with interior point algorithm.



- Allowed and disallowed $(\Delta_\epsilon - \Delta_\sigma)$ points in a Z_2-symmetric CFT3 with only one relevant Z_2-odd and Z_2-even scalar.
- The light grey points are ruled out, while the dark blue points are allowed. The light blue shaded region shows the region allowed by crossing symmetry and unitarity of the single correlator $\langle\sigma\sigma\sigma\sigma\rangle$.The final allowed region is the intersection of this shaded region with the region indicated by the dark blue points.

- Improve the accuracy
  - Using different algorithms, like Second Order Conic Programming (SOCP), cutting plane methods, or constrained nonlinear optimization and so on.
- Using new constraints
  - crossing symmetry for another correlators such as strees tensor operator and so on.

- Stress tensor or global symmetry current as external operator
  - The situation is more complicated.
  - There are different four-point structures.
  - Different structures appear in three-point functions containing two stress tensor and so on.
  - We use the embedding space formalism to overcome to the above difficulties.

Thank you