Entanglement Entropy of Interacting QFTs

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Outline

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   - Area Law Divergence
2. EE of two QFTs
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3. Holographic EE of Interacting CFTs
   - Holographic Entanglement Entropy
   - A Holographic Setup
   - Generalized HEE

Definition of Entanglement Entropy (EE):

- Consider a Hilbert space factorization as:

$$\mathcal{H}_{\rm tot} = \mathcal{H}_A \otimes \mathcal{H}_B$$

- Reduced density matrix (for part A)

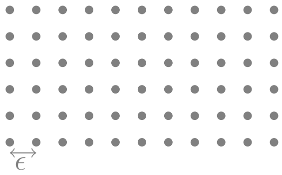
$$\rho_A = {\rm Tr}_B \left[\rho_{\rm tot}\right]$$

- EE: von-Neumann entropy for ρ_A

$$S_A = -{\rm Tr}_A \left[\rho_A \log \rho_A\right]$$

EE in QFTs:

- Consider a QFT defined on $\mathbb{R} \times \mathcal{M}$,
- Divide $\mathcal{M}$ to A and B such that $A \cup B = \mathcal{M}$,



Many-body System

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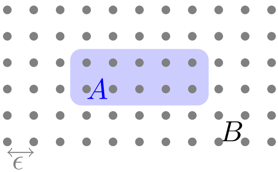

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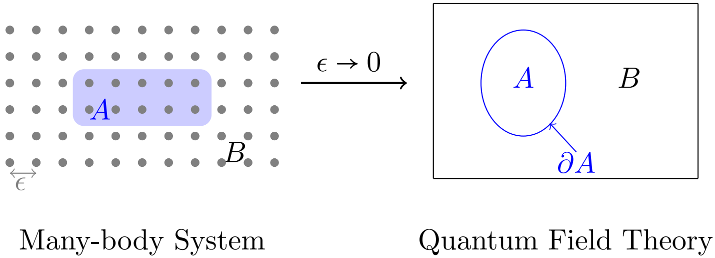


EE Properties

- For a pure ρ_{tot} we find $S_A = S_B$
  EE is not extensive

- For non-intersecting subsystems A, B, C

$$\begin{aligned} S_{A+B+C} + S_B &\leq S_{A+B} + S_{B+C} \\ S_A + S_C &\leq S_{A+B} + S_{B+C} \end{aligned}$$

  Strong Subadditivity

- If $B = \varnothing$, SS reduces to subadditivity

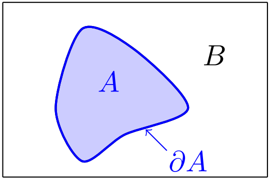
$$S_{A+B} \leq S_A + S_B$$

Area Divergence (Area Law)

- In a QFT, EE suffers from UV divergence
  [Bombelli-Koul-Lee-Sorkin 86, Srednicki 93]
- EE in a $(d+1)$ dim. **local** QFT is proportional to the $(d-1)$ dim boundary ∂A

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{\epsilon^{d-1}} + \cdots$$

- Intuitively the most strongly entangled region lies around ∂A



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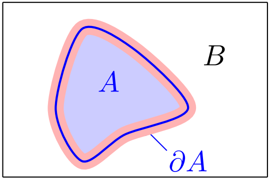

Area Divergence (Area Law)

- A special case: 2-dim CFT

$$S_A = \frac{c}{3} \log \frac{\ell}{\epsilon}$$

- For $(d+1)$-dim CFTs, for a smooth entangling region (calculated by HEE) [Ryu-Takayanagi 06]

$$S_A = \gamma_1 \cdot \left(\frac{\ell}{\epsilon}\right)^{d-1} + \gamma_3 \cdot \left(\frac{\ell}{\epsilon}\right)^{d-3} + \cdots + \begin{cases} \gamma_{d-1} \cdot \left(\frac{\ell}{\epsilon}\right) + \gamma_d & d : \text{even} \\ \gamma_{d-2} \cdot \left(\frac{\ell}{\epsilon}\right)^2 + \tilde{c} \log \frac{\ell}{\epsilon} & d : \text{odd} \end{cases}$$

- For $d = 1$, $\tilde{c} = c/3$
- For $d = 3$ the result is confirmed in CFT_4

Area Law Violation

1. Logarithmic Divergence [Wolf 05, Gioev-Klich 05, Ogawa-Takayanagi-Ugajin 11, Huijse-Sachdev-Swingle 11]
   - In systems with Fermi surface

$$S_A = c_1 \left(\frac{L}{\epsilon}\right)^{d-1} + c_2 \left(L k_F\right)^{d-1} \log\left(\ell k_F\right) + \mathcal{O}\left(\ell^0\right)$$

   - For $k_F \sim \epsilon^{-1}$, if $\ell \sim L$, the second term is leading!

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2. Volume Divergence [Shiba-Takayanagi 13, Karczmarek-Sabella-Garnier 13]
   - Volume law divergence has been in a non-local QFT defined (in 1+1 dim.) by

$$H = \int dx \left[\dot{\phi}^2(x) + \phi(x) e^{\alpha\sqrt{-\partial^2}} \phi(x)\right]$$

   - For $\ell \ll \alpha$, volume law has been observed i.e. $S_A \sim \ell\alpha$

Entanglement Entropy of two QFTs

Entanglement via Interaction

- Two QFTs living on a common spacetime

$$S = \int dx^d \left[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{int}\right]$$

- Hilbert space decomposition

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Trace out ρ_{tot} over either QFTs:

$$S_{ent} = -\mathrm{Tr}\left[\rho_1 \log \rho_1\right], \quad \rho_1 = \mathrm{Tr}_{\mathcal{H}_2}[\rho_{tot}].$$

- If $\mathcal{L}_{int} = 0$, S_{ent} vanishes.

[A. M., N. Shiba, T. Takayanagi JHEP 04(2014)185]

Massive and Massless Models

- Massless interaction

$$S = \frac{1}{2}\int d^d x[(\partial_\mu\phi)^2 + (\partial_\mu\psi)^2 + \lambda\partial_\mu\phi\partial^\mu\psi].$$

- Massive interaction

$$S = \frac{1}{2}\int d^d x\left[(\partial_\mu\phi)^2 + (\partial_\mu\psi)^2 - (\phi,\psi)\begin{pmatrix} A & C \\ C & B \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}\right],$$

- Trace out ϕ and calculate EE of ψ

$$\rho_\psi[\psi_1,\psi_2] = \int D\phi\ \Psi^*[\phi,\psi_1]\Psi[\phi,\psi_2]$$

Field Theory Results

Replica wave functional method leads to:

1. Massless case

$$S_\psi \propto V_{d-1} \Lambda^{d-1}$$

Volume law divergence

2. Massive case

$$\frac{S_\psi}{V_{d-1}} \propto \begin{cases} \text{finite} & for \quad d \leq 4 \\ \frac{1}{2} (\ln \Lambda)^2 & for \quad d = 5 \\ \frac{1}{d-5} \Lambda^{d-5} \ln \Lambda & for \quad d \geq 6. \end{cases}$$

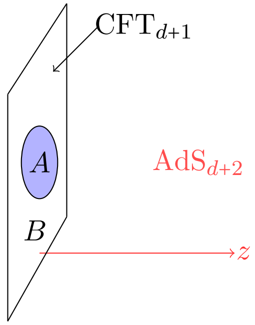
Suppression of volume law divergence

Holographic EE of Interacting CFTs

Holographic Entanglement Entropy

- In the context of AdS/CFT correspondence, Ryu and Takayanagi proposed (2006):

$$\boxed{S_A = \frac{1}{4G_N^{(d+2)}} \min_{\partial A = \partial \gamma_A} [\text{Area}(\gamma_A)]}$$



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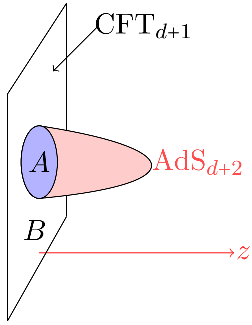

Gravity Dual of Two Interacting CFTs

- Consider N D3-brane located at $\vec{y}_a$, $a = 1, 2, \cdots, N$
- Type IIB SUGRA solution (in the NHL)

$$ds^2 = f^{-1/2}(\vec{y}) dx^\mu dx_\mu + f^{1/2}(\vec{y}) dy^i dy^i$$

where $\mu = 0, 1, 2, 3$ and $i = 1, 2, \cdots, 6$

$$f(\vec{y}) = \sum_{a=1}^{N} \frac{R^4}{|\vec{y} - \vec{y}_a|^4}$$

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- If all D3-branes are located at one point ($|y| = r$)

$$ds^2 = \frac{r^2}{R^2} dx^\mu dx_\mu + \frac{R^2}{r^2} \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\Omega_4^2 \right)$$

which is $\mathrm{AdS}_5 \times \mathrm{S}^5$

SU(N/2)×SU(N/2) Solution

- Consider two stack of n and m D3-branes at $\vec{Y}_{1,2} = (\pm\ell, \vec{0})$

$$f(\vec{y}) = \frac{n}{N}\frac{R^4}{|\vec{y}-\vec{Y}_1|^4} + \frac{m}{N}\frac{R^4}{|\vec{y}-\vec{Y}_2|^4}$$

- For $n = m = \frac{N}{2}$, and $\Lambda \ll \ell$, the gauge theory sym. is

$$\mathrm{SU}(N/2)\times\mathrm{SU}(N/2)$$

- **Argument:** EE between these CFTs is given by

$$S_{\text{ent}} = \frac{\text{Area}(\gamma)}{4G_N^{10}},$$

γ: Minimal surface separating the two stacks of branes

HEE of Two SU($N/2$) Gauge Theories

- γ is 8-dim surface:

$$\gamma : t = 0 \ , \ \ \theta = \frac{\pi}{2}$$

-

$$S_{\rm ent} = \frac{V_3 {\rm Vol}(S^4)}{4G_N^{(10)}} \int_0^{r_{UV}} R^2 \frac{r^4}{r^2 + \ell^2} dr$$

- With $r_{UV} = \Lambda R^2$ and $r_{UV} \ll \ell$

$$S_{\rm ent} \simeq \frac{16 N^2 V_3}{15\pi^3} \lambda g \Lambda^3$$

- The same result from D3-brane shell solution

A proposal for generalized HEE

- In classical gravity approx. we have a $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ setup
- The gravity dual is described by $\mathrm{M}_{q+d+1} = \mathrm{Y}^{AdS}_{d+1} \times \mathrm{X}_q$

$$\mathrm{Y}^{AdS}_{d+1} : (d+1) \text{ dimensional asymptotically AdS space}$$
$$\mathrm{X}_q : q \text{ dimensional internal space}$$

- In Poincare coor. $\partial \mathrm{M}_{q+d+1} = \mathbb{R}^{1,d-1} \times \mathrm{X}_q$
- Consider A and B such that $\partial A (= \partial B)$ divides $\mathbb{R}^{1,d-1} \times \mathrm{X}_q$ into two parts at $t = t_0$
- Main assumption: separation of time slices of $\partial \mathrm{M}_{q+d+1}$ corresponds to a factorization of Hilbert space in the dual CFT:

$$\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_B .$$

A proposal for generalized HEE

- Under this assumption, EE of A is given by

$$S_{\text{ent}} = \frac{\text{Area}(\gamma)}{4G_N},$$

  γ: the minimal (or extremal in time-dependent cases) surface i.e. $\partial\gamma = \partial A$
- In particular, if ∂A wraps X_q completely, this prescription is reduced to the standard HEE

Examples of GHEE

- A fundamental example: $\mathrm{AdS}_5\times\mathrm{S}^5$
  $A(B)$: Northern (southern) hemisphere of S^5

$$S_{\mathrm{ent}} = \frac{4}{9\pi^2}\frac{N^2 V_3}{\epsilon^3}$$

$S_{\mathrm{ent}} < \frac{1}{2\pi}\frac{N^2 V_3}{\epsilon^3} (= S_{\mathrm{max}})$ [Susskind & Witten 98]

- Finite temp. case: $\mathrm{AdS}_5\text{-Schld}\times\mathrm{S}^5$

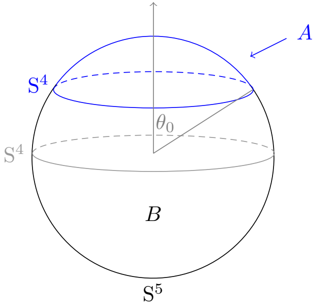
$$S_{\mathrm{ent}} = \frac{4}{9\pi^2}\frac{N^2 V_3}{\epsilon^3} + 1.83\cdot N^2 V_3 T^3$$

comparable with free $SU(N/2)$ $\mathcal{N}=4$ SYM [RT 06]

$$S_{\mathrm{thermal}}^{\mathrm{free}} \simeq 1.64\cdot N^2 V_3 T^3$$

Examples of GHEE

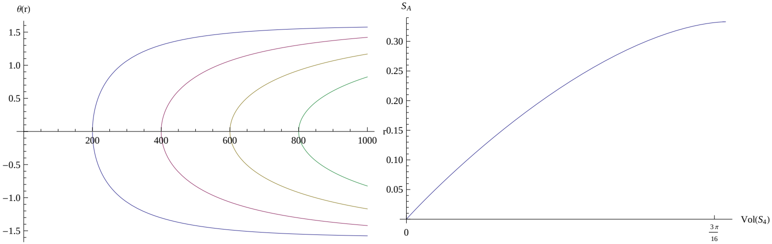
- If A is not half of S^5 (∂A: $\theta = \theta_0$ at $t = 0$)



- Entanglement of SU(n) and SU(m) subsectors of SU(N)

$$S_{\rm ent} = \frac{R^2 V_3 \mathrm{Vol}(S^4)}{4G_N^{(10)}} \int_{r_*}^{r_{UV}} r^2 \sin^4\theta(r)\sqrt{1 + r^2\dot{\theta}^2(r)}dr$$

Examples of GHEE



$$m = \frac{8}{3\pi} \cdot (n + m) \int_0^{\theta_0} d\theta \sin^4 \theta$$

Summery & Conclusions

- EE of two interacting CFTs → volume law divergence (suppressed in massive QFTs)
- Holographically dividing the internal space could address the EE of two interacting CFTs
- GHEE: Minimize a surface in the whole manifold $\mathrm{M}_{q+d+1} = \mathrm{Y}^{AdS}_{d+1} \times \mathrm{X}_q$ rather than only the asymptotically AdS part Y^{AdS}_{d+1}