Entanglement and Symmetry

Amin Faraji Astaneh

In Collaboration with Gary Gibbons and Sergey Solodukhin

IPM

hep-th/1407.4719

March 2, 2015

Overview

1. EE in QM
2. EE in QFT
3. Holographic Entanglement Entropy
4. What surface maximizes EE?

EE in QM

- Consider a quantum mechanical system in a pure ground state which is described by $|\psi\rangle$ ($\rho = |\psi\rangle\langle\psi|$).

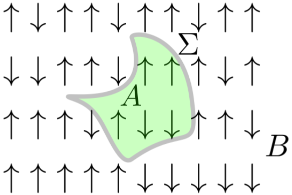


Figure : Note: Σ is imaginary!

- *Reduced* density operator:

$$\rho_A = \mathrm{Tr}_B\, \rho = \mathrm{Tr}_B |\psi\rangle\langle\psi|\,.$$

Then the EE is

$$S_{EE}(A) = -\,\mathrm{Tr}\rho_A \log \rho_A\,.$$

EE in QFT

$$S_{EE}(A) \geq 0\,.$$

$$S_{EE}(A) = S_{EE}(A^c)\,.$$

$$\ldots$$

Area law [Srednicki (03)]

$$S_{EE} \sim \frac{\mathcal{A}(\Sigma)}{\epsilon^{d-2}}\,.$$

Holographic EE

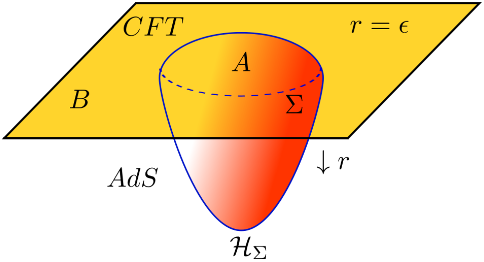
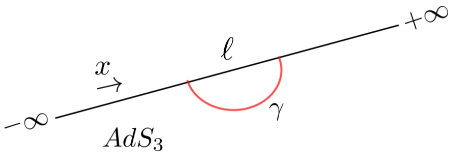


Figure : Ryu-Takayanagi's (RT) proposal (06)

$$S_{HE}(\Sigma) = \text{Min}\,\frac{\mathcal{A}(\mathcal{H}_\Sigma)}{4G_N^{(d+1)}}\,,$$

Holographic EE for a $1+1$ dimensional CFT



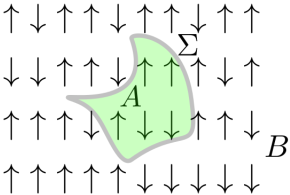
In the case of a $1+1$ dimensional CFT we should calculate the geodesic length, γ, in AdS_3 described by

$$ds^2 = \frac{R^2}{r^2}(-dt^2 + dx^2 + dr^2)\,,$$

then using RT proposal the EE reads

$$S_{HE}(\gamma) = \frac{R}{2G_N}\log\frac{\ell}{\epsilon} = \frac{c}{3}\log\frac{\ell}{\epsilon}\,.$$

What surface maximizes EE?



To give an answer we used the RT prescription in an asymptotic form, this leads to C. R. Graham, E. Witten (99), A.F.A, G.Gibbons, S.Solodukhin(14)

$$dv_{\mathcal{H}_\Sigma} = r^{-d+1}\left[1 - \frac{1}{2}\left(\frac{d-3}{(d-2)^2}(\,\mathrm{Tr}K)^2 + \,\mathrm{Tr}P\right) r^2 + \cdots\right] dv_\Sigma \; dr \;,$$

where,

$$P_{\alpha\beta} = \frac{1}{d-2}\left(R_{\alpha\beta} - \frac{R}{2(d-1)} g_{\alpha\beta}\right) \;.$$

Entanglement between Math and Physics!

$$S_{HE}(\Sigma) = \frac{A(\mathcal{H}_\Sigma)}{4G_N} = \frac{1}{4G_N}\frac{A(\Sigma)}{(d-2)\epsilon^{d-2}} +$$
$$+ \frac{1}{4G_N}\frac{1}{2(d-2)(d-4)\epsilon^{d-4}}\int_\Sigma dv_\Sigma$$
$$\left[R_{aa} - \frac{d}{2(d-1)}R - \frac{d-3}{d-2}(\mathrm{Tr}K)^2\right] .$$

The question is: what surface minimizes the Willmore functional?

$$W(\Sigma) = \frac{1}{4}\int_\Sigma (\mathrm{Tr}K)^2 \, ,$$

Max of $S_{EE}(\Sigma)$ when $\mathcal{A}(\Sigma)$ is fixed $\sim$ Min of $W(\Sigma)$.

Formulation of the problem

Consider a field theory defined on the Minkowski spacetime. Let us also consider that the area of the entangling region $\mathcal{A}(\Sigma)$ is fixed. We are looking for

- The miximizer of the entropy (minimizer of the Willmore functional), Σ_0, when topology is fixed such that

$$S(\Sigma) \leq S(\Sigma_0) \quad \textit{topology is fixed}$$

- The global maximizer of the entropy (minimizer of the Willmore functional), Σ_m, for any surface Σ of same area A and arbitrary topology in any dimension

$$S(\Sigma) \leq S(\Sigma_m) \quad \textit{any topology}$$

Our observation/conjecture is:

A.F.A, G.Gibbons, S.Solodukhin(14)

first) Round sphere, S^{d-2}, is the entropy maximizer in its own topology class.

then) Round sphere is the entropy global maximizer for other hyper surfaces with the same area but different topology.

Lets check it in $4D$.

Entropy maximizer in $4D$ is S^2

$$W(\Sigma) = \frac{1}{4}\int_\Sigma (\,\mathrm{Tr}K)^2 \,.$$

Doing some rewriting we get

$$\frac{1}{2}(\,\mathrm{Tr}K)^2 = R_\Sigma + K_\Sigma \,,$$

$$R_\Sigma = (\,\mathrm{Tr}K)^2 - \,\mathrm{Tr}K^2 \,, \, K_\Sigma = \,\mathrm{Tr}K^2 - \frac{1}{2}(\,\mathrm{Tr}K)^2 \,,$$

but

$$K_\Sigma = (K_{ij} - \frac{1}{2}\gamma_{ij}\,\mathrm{Tr}K)^2 \,,$$

demanding $K_\Sigma = 0 \to K_{ij} = \frac{1}{2}\gamma_{ij}\,\mathrm{Tr}K$. Using the Gauss-Codazzi equations

$$\nabla^j K_{ij} = \nabla_i\,\mathrm{Tr}K \to \,\mathrm{Tr}K = const.$$

$\therefore \; R_\Sigma = const. \geq 0 \to \Sigma_0 \sim S^2.$

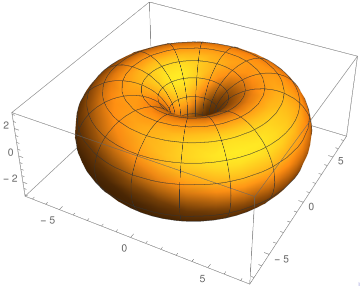
Maximizers of entropy, Willmore conjecture

We Proved (in $4D$) and Conjectured (in higher D) that the Spheres are the global maximizers of the EE.

$$g = 0 \rightarrow W(\Sigma) \geq W(S^2) = 4\pi$$
$$g = 1 \rightarrow W(\Sigma) \geq W(\mathbf{T}^2_{cliff}) = 2\pi^2 \ .$$

In $g = 1$ class Clifford torus is the entropy maximizer.

$$S_{EE}(\Sigma_{g=1}) \leq S_{EE}(\mathbf{T}^2_{Cliff})$$



Renormalized Willmore functional in higher D

Lets define the renormalized Willmore energy as

$$\widehat{W}(\Sigma_{d-2}) = W(\Sigma_{d-2})/A^{\frac{d-4}{d-2}}\,,\;\; W(\Sigma_{d-2}) = \frac{1}{4}\int_{\Sigma_{d-2}} (\,\mathrm{Tr}K)^2\,.$$

For example for the round sphere

$$\widehat{W}(S^{d-2}) = \frac{W(S^{d-2})}{[A(S^{d-2})]^{\frac{d-4}{d-2}}} = \frac{(d-2)^2}{4}\left(\frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\right)^{\frac{2}{d-2}}\,.$$

Who wins the game?

Maximizers of entropy, Ellipsoid in Higher D

Consider the ellipsoid E^{d-2}

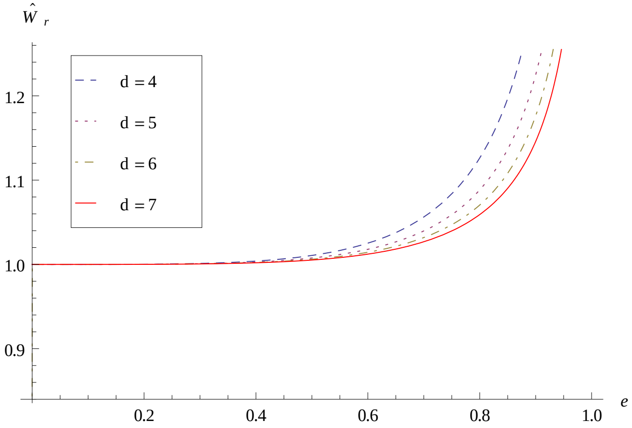
$$\frac{x_1^2}{a_1^2} + \ldots + \frac{x_{d-1}^2}{a_{d-1}^2} = 1\,,\; (a_1 = a_2 = \cdots = a_{d-2} = a) \neq (a_{d-1} = b)\,.$$

The desired quantity is

$$\widehat{W}_r(e) = \frac{\widehat{W}(E^{d-2})}{\widehat{W}(S^{d-2})}\,, e = \sqrt{1 - \frac{a^2}{b^2}}\,.$$

$$\begin{aligned} W(E^{d-2}) =& \frac{1}{4}\frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} a^{d-4}(1-e^2)^{\frac{d-4}{2}} \Bigg[{}_2F_1(\frac{d-2}{2}, \frac{d-6}{2}, \frac{d-1}{2}, e^2) \\ & + (d-3)^2 \; {}_2F_1(\frac{d-2}{2}, \frac{d-2}{2}, \frac{d-1}{2}, e^2) \\ & + 2(d-3) \; {}_2F_1(\frac{d-2}{2}, \frac{d-4}{2}, \frac{d-1}{2}, e^2) \Bigg]\,. \end{aligned} \tag{1}$$

Maximizers of entropy, Ellipsoid in Higher D



$$S_{EE}(E^{d-2}) \leq S_{EE}(S^{d-2})\,.$$

Maximizers of entropy, $S^m \times S^n$ geometry in Higher D

Let us consider a toric geometry in higher dimensions

$$\widehat{W}_r(x) = \frac{\widehat{W}(S^m \times S^n)}{\widehat{W}(S^{m+n})}\,, x = \frac{r}{R}\,.$$

What happens?

Maximizers of entropy, $S^m \times S^n$ geometry in Higher D

As a generalization of a torus, for $S^m \times S^n$ geometries we have

- $d = m + n + 2 = 4$

d=4	$S^1 \times S^1$
x_{min}	0.707
$\widehat{W}_{r,min}$	1.571

- $d = m + n + 2 = 5$

d=5	$S^2 \times S^1$	$S^1 \times S^2$
x_{min}	0.886	0.816
$\widehat{W}_{r,min}$	1.391	1.333

- $d = m + n + 2 = 6$

d=6	$S^3 \times S^1$	$S^2 \times S^2$	$S^1 \times S^3$
x_{min}	0.968	1	1
$\widehat{W}_{r,min}$	1.324	1.237	1.116

- $d = m + n + 2 = 7$

d=7	$S^4 \times S^1$	$S^3 \times S^2$	$S^2 \times S^3$	$S^1 \times S^4$
x_{min}	0.9987	1	1	1
$\widehat{W}_{r,min}$	1.289	1.226	1.152	1.076

Conclusion

Most symmetric is the most entropic!