Holographic Entanglement Entropy

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IPM School and Workshop on Applied AdS/CFT

Feb 28- March 4, 2015, Tehran

Plan

- Review of (holographic) entanglement entropy

- Explicit example: static and time dependent holographic entanglement entropy

- Special topics: different lines of research

References

P. Calabrese and J. Cardy, “Entanglement entropy and conformal field theory,” arXiv:0905.4013.

T. Nishioka, S. Ryu and T. Takayanagi, “Holographic Entanglement Entropy: An Overview,” arXiv: 0905.0932.

S. N. Solodukhin, “Entanglement entropy of black holes,” arXiv:1104.3712

H. Liu and S. J. Suh, “Entanglement growth during thermalization in holographic systems,” arXiv:1311.1200

Entanglement entropy

Consider a state $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, which evolves in time by its Hamiltonian H

Physical quantities are computed as expectation values of operators as follows

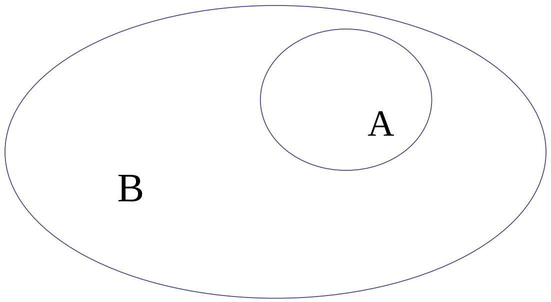
$$\langle O \rangle = \langle \psi | O | \psi \rangle = \mathrm{Tr}(\rho O)$$

where we defined the density matrix $\rho = |\psi\rangle\langle\psi|$. This system is called a pure state as it is described by a unique wave function $|\psi\rangle$.

In mixed states, the system is described by a density matrix ρ. An example of a mixed state is the canonical distribution

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$$

Assume that the quantum system has multiple degrees of freedom and so one can decompose the total system into two subsystems A and B



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

The reduced density matrix of the subsystem A

$$\rho_A = \mathrm{Tr}_B(\rho)$$

Then the entanglement entropy is defined as the von-Neumann entropy

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

A measure how much a given quantum state is quantum mechanically entangled.

An example

Consider a system consisting of two spinors with $s = \frac{1}{2}$. Consider the following state

$$|\psi\rangle = \sin\theta |\uparrow\rangle_A |\downarrow\rangle_B + \cos\theta |\downarrow\rangle_A |\uparrow\rangle_B \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

Using the density matrix $\rho = |\psi\rangle\langle\psi|$ one has

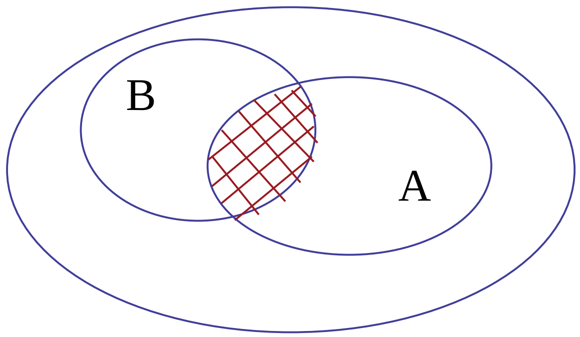
$$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \sin^2\theta |\uparrow\rangle_A |\uparrow\rangle_A + \cos^2\theta |\downarrow\rangle_A |\downarrow\rangle_A$$

Therefore

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A) = -\sin^2\theta \ln \sin^2\theta - \cos^2\theta \ln \cos^2\theta$$

$$\theta = 0, \frac{\pi}{2} \longrightarrow S_A = 0,$$
$$\theta = \frac{\pi}{4} \longrightarrow S_A = \ln 2$$

Properties of Entanglement entropy



1. $S_A \geq 0$ and $S_A \leq \ln(\mathrm{Dim}\{\mathrm{Hibert}\})$.

1. For pure state $S_A = S_B$

2. For two subspace A and B, the strong subadditivity is

$$S_A + S_B \leq S_{A\cup B} + S_{A\cap B}$$

3. Leading divergence term is proportional to the area of the boundary ∂A

$$S_A = c_0 \frac{\mathrm{Area}(\partial A)}{\epsilon^{d-1}} + O(\epsilon^{-(d-2)}),$$

Rényi entropies

It is also useful to compute Rényi entropies

$$S_n = \frac{1}{1-n} \log \mathrm{Tr} \rho^n$$

Then the entanglement entropy is given by

$$S_E = \lim_{n \to 1} S_n$$

Practically one may first compute $\mathrm{Tr}(\rho^n)$ by making use the replica trick and then

$$S_E = -\partial_n \mathrm{Tr} \rho^n |_{n=1}$$

Consider a quantum field theory with a generic field $\phi(\tau, \vec{x})$. The amplitude to go from a field configuration ϕ_1 at τ_1 to a field configuration ϕ_2 at τ_2

$$\langle\phi_2(\tau_2, \vec{x})|\phi_1(\tau_1, \vec{x})\rangle = \langle\phi_2|e^{-H(\tau_2-\tau_1)}|\phi_1\rangle = N \int \mathcal{D}\phi \; e^{-S}$$

The element of the density matrix is

$$\rho_{\phi_2\phi_1} = \langle\phi_2|e^{-H(\tau_2-\tau_1)}|\phi_1\rangle$$

Let us assume a periodic boundary condition

$$\phi_2(\tau_2, \vec{x}) = \phi_1(\tau_1, \vec{x}) \qquad \text{for} \quad \tau_2 = \tau_1 + 2\pi\beta$$

.

The trace of density matrix is given by

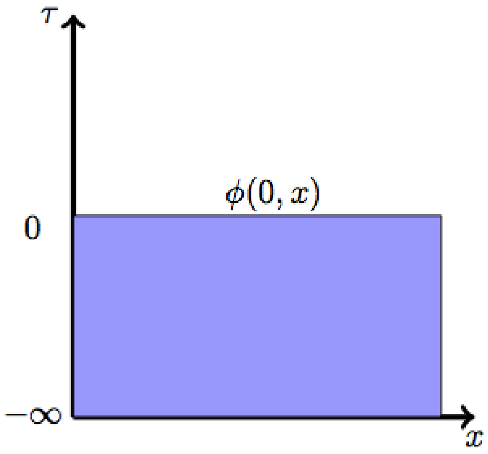
$$\mathrm{Tr}(\rho) = \int d\phi_1 \langle \phi_1 | e^{-H\beta} | \phi_1 \rangle = N \int_{\text{periodic}} \mathcal{D}\phi \; e^{-S}$$

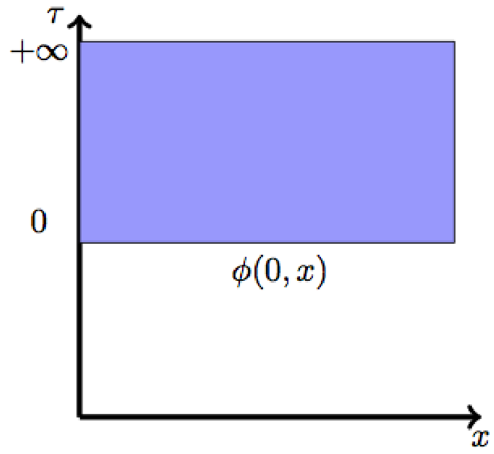
This is indeed the definition of partition function. Therefore one has

$$\mathrm{Tr}(\rho) = Z[\beta]$$

The reduced density matrix should be also given in terms of

"partition function".





Consider a field configuration ϕ at $\tau = 0$. Then the corresponding wave function is

$$\Psi[\phi] = \int_{\phi(-\infty)=0}^{\phi(0)} \mathcal{D}\phi \; e^{-S}$$

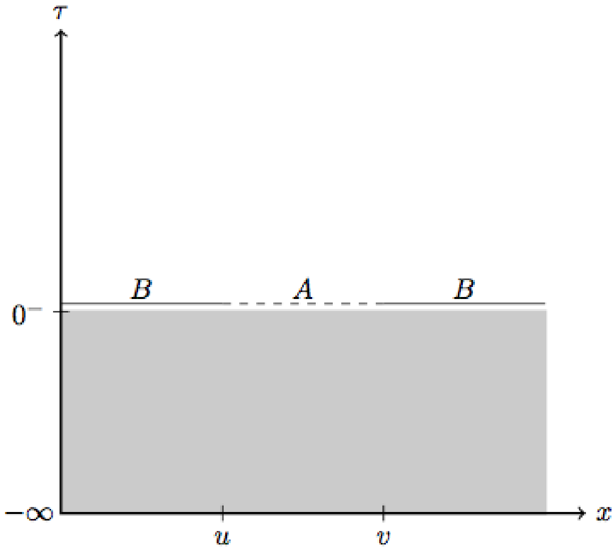
Similarly

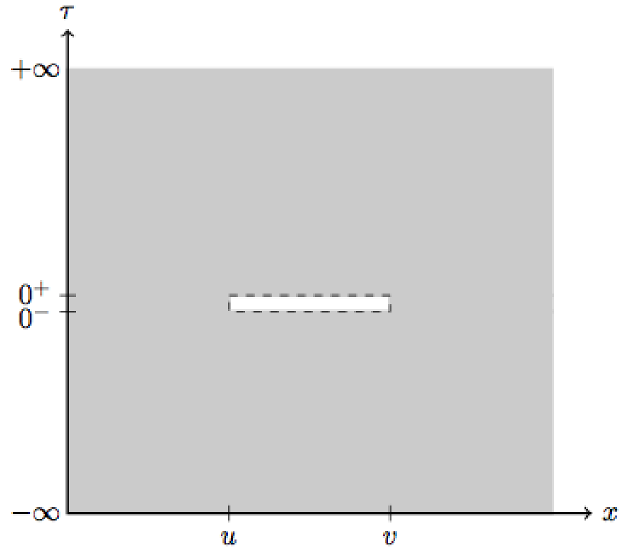
$$\Psi^*[\phi] = \int_{\phi(0)}^{\phi(\infty)=0} \mathcal{D}\phi \; e^{-S}$$

So that

$$\rho_{\phi_2\phi_1} = \Psi^*[\phi_2]\Psi[\phi_1]$$

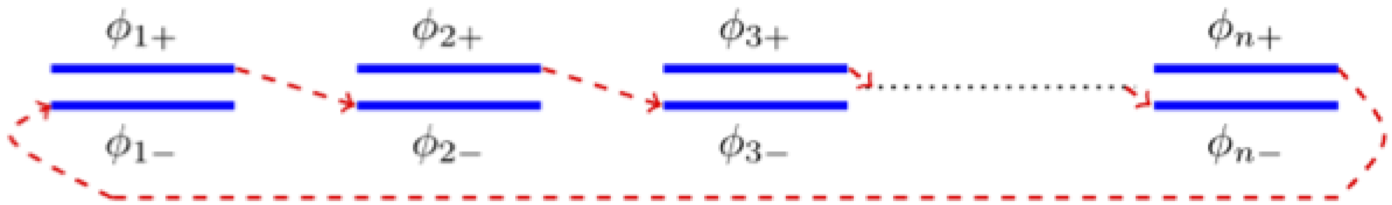
Reduced density matrix





$A : x \in [u, v]$ $\qquad$ $B : x \in (-\infty, u] \cup [v, \infty)$

$$[\rho_A]_{\phi^+\phi^-} = N \int_{\phi(-\infty)=0}^{\phi(\infty)=0} \mathcal{D}\phi \; e^{-S} \prod_{x \in A} \delta(\phi(0^-) - \phi^-)\delta(\phi(0^+) - \phi^+)$$



The nth power of reduced density matrix is

$$[\rho_A]_{\phi_1^+\phi_1^-}[\rho_A]_{\phi_2^+\phi_2^-}[\rho_A]_{\phi_3^+\phi_3^-}\cdots[\rho_A]_{\phi_n^+\phi_n^-}$$

With the identification

$$\phi_i^- = \phi_{i+1}^+$$

The trace is then given by further identification

$$\phi_n^- = \phi_1^+$$

$$\mathrm{Tr}(\rho^n) = [\rho_A]_{\phi_n^+\phi_1^-}[\rho_A]_{\phi_1^+\phi_2^-}[\rho_A]_{\phi_2^+\phi_3^-}\cdots[\rho_A]_{\phi_{n-1}^+\phi_n^-}$$

Using the definition of reduced density matrix one arrives at

$$\mathrm{Tr}(\rho^n) = N^n \int_{M_n} \mathcal{D}\phi \, e^{-S} \equiv N^n Z_n$$

Thus

$$\begin{aligned} S &= -\partial_n \log \mathrm{Tr}(\rho^n)|_{n=1} = -\partial_n (\log N^n Z_n) \\ &= -\partial_n \Big(\log Z_n - n \log Z_1 \Big)\Big|_{n=1} \end{aligned}$$

Here, as usual, we set $N = Z_1^{-1}$.

AdS/CFT correspondence

Basically AdS/CFT correspondence is a duality or a relation between two theories one with a gravity and the other without gravity.

The gravitational theory is usually defined in higher dimension.

Well developed case is the one where the gravity is defined on an AdS geometry where the dual theory is a CFT living in the conformal boundary of AdS space.

Classical gravity on an asymptotically locally AdS$_{d+1}$ background is dual to a d-dimensional Large N strongly coupled field theory with a UV fixed point on its boundary.

AdS$_{d+1}$ metric in Poincare coordinates

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2.$$

AdS$_{d+1}$ metric in global coordinates

$$ds^2 = -(1 + \frac{r^2}{R^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega_{d-1}^2.$$

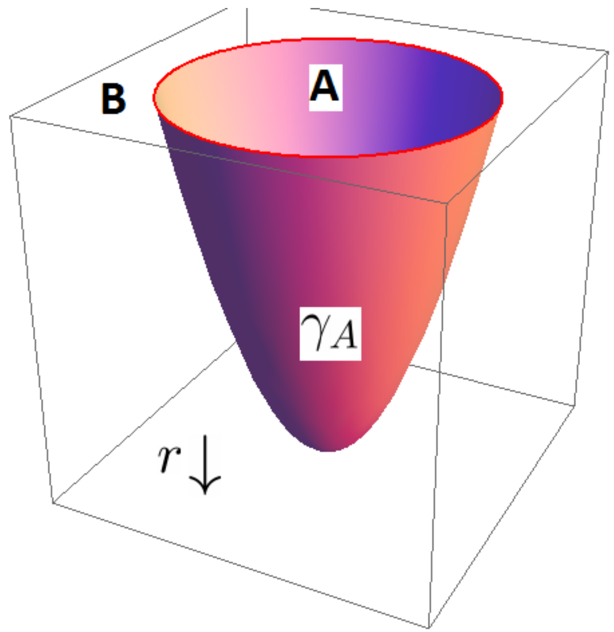
Here boundary is at $r \to \infty$

There is one to one correspondence between objects in CFT and those in the gravitational theory on AdS space.

Gravity	$\Longleftrightarrow$	Field theory
$\{r, t, \vec{x}\}$	$\Longleftrightarrow$	$\{$scale of energy$, t, \vec{x}\}$
Near boundary Near horizon	$\Longleftrightarrow$	UV, IR regions
Symmetries	$\Longleftrightarrow$	Symmetries
Fields $\Phi(r, t, \vec{x})$	$\Longleftrightarrow$	Operators $\mathcal{O}(t, \vec{x})$
On shell action	$\Longleftrightarrow$	Generating function

Holographic Formula for Entanglement Entropy

For static background and fixed time divide the boundary into A and B. Extend this division $A \cup B$ to of the bulk spacetime. Extend ∂A to a surface γ_A in the entire spacetime such that $\partial \gamma_A = \partial A$.



$$S_A = \frac{\text{Area}(\gamma_A)}{4 G_N^{(d+2)}}$$

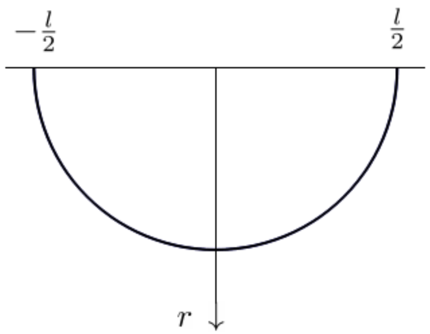
Static solutions

Let's compute the holographic entanglement entropy for a strip in a static asymptotically AdS geometry.

$$dS^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + g(r)dr^2 + dx_1^2 + dx_{d-2}^2\right),$$

For black hole solution

$$f(r) = g(r)^{-1} = 1 - mr^d = 1 - \frac{r^d}{r_H^d}$$



Consider an entangling region in the shape of a strip with the width of ℓ given by

$$-\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}, \qquad 0 \leq x_i \leq L, \qquad i = 2, \cdots, d-2.$$

The holographic entanglement entropy may be computed by minimizing a codimension two hypersurface in the bulk geometry whose intersection with the boundary coincides with the above strip.

Assuming that the profile of the hypersurface in the bulk is parameterized by $x_1 = x(r)$, the corresponding induced metric is

$$dS^2_{\text{ind}} = \frac{R^2}{r^2}\left[\left(g(r) + x'^2\right) dr^2 + d\vec{x}^2\right].$$

Therefore the area A reads

$$A = L^{d-2} R^{d-1} \int dr \frac{\sqrt{g + x'^2}}{r^{d-1}},$$

$$\frac{\ell}{2} = \int_0^{r_t} dr \frac{\sqrt{g(r)}\left(\frac{r}{r_t}\right)^{d-1}}{\sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}}, \qquad S = \frac{L^{d-2} R^{d-1}}{2G_N} \int_\epsilon^{r_t} \frac{\sqrt{g(r)}dr}{r^{d-1}\sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}}$$

where r_t is a turning point and ϵ is a UV cut-off.

For the vacuum state where $f(r) = g(r) = 1$ (AdS solution) one gets

$$S = \begin{cases} \frac{L^{d-2}R^{d-1}}{2G}\left(-\frac{1}{(d-1)\epsilon^{d-2}} + \frac{c_0}{\ell^{d-2}}\right) & \text{for } d \neq 2, \\ \\ \frac{R}{2G}\ln\frac{\ell}{\epsilon} = \frac{c}{3}\ln\frac{\ell}{\epsilon}, & \text{for } d = 2, \end{cases}$$

with c_0 being a numerical factor

$$c_0 = \frac{2^{d-2}\pi^{\frac{d-1}{2}}}{d-2}\left(\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}\right)^{d-1}$$

When $f \neq 1$, in general, it is not possible to find an explicit expression for the entanglement entropy, though in certain limits one may extract a general behavior of the entanglement entropy.

In particular in the limit of $ml^d \ll 1$, one finds

$$\Delta A = \frac{L^{d-2}}{2} \int d\rho \ \ \delta_f \left(\frac{\sqrt{f^{-1} + x'^2}}{\rho^{d-1}} \right) \Big|_{f=1} \Delta f,$$

which leads to the following expression for the entanglement entropy

$$S_{\mathrm{BH}} = S_{\mathrm{vac}} + \frac{L^{d-2}}{4G_N} c_1 m \ell^2,$$

where S_{vac} is the entanglement entropy of the vacuum solution, and

$$c_1 = \frac{1}{16(d+1)\sqrt{\pi}} \frac{\Gamma(\frac{1}{2(d-1)})^2 \Gamma(\frac{1}{d-1})}{\Gamma(\frac{d}{2(d-1)})^2 \Gamma(\frac{1}{2} + \frac{1}{d-1})}.$$

For $m\ell^d \gg 1$ the main contributions to the entanglement entropy comes from the limit where the minimal surface is extended all the way to the horizon so that $\rho_t \sim \rho_H$. Setting $u = \frac{\rho}{\rho_t}$ one gets

$$\frac{\ell}{2} \approx \rho_H \int_0^1 \frac{u^{d-1}du}{\sqrt{(1-u^d)\left(1-u^{2(d-1)}\right)}},$$

$$S_{\text{BH}} \approx \frac{L^{d-2}}{4G_N\rho_H^{d-2}} \int_{\frac{\epsilon}{\rho_H}}^1 \frac{du}{u^{d-1}\sqrt{(1-u^d)\left(1-u^{2(d-1)}\right)}}.$$

Note that apart from the UV divergent term in S_{BH}, due to the double zero in the square roots, the main contributions in the above integrals come from $u = 1$ point. Indeed around $u = 1$ it may be recast to the following form

$$S_{\text{BH}} \approx \frac{L^{d-2}}{4G_N\rho_H^{d-2}} \left(\int_0^1 \frac{u^{d-1}du}{\sqrt{(1-u^d)\left(1-u^{2(d-1)}\right)}} + \int_{\frac{\epsilon}{\rho_H}}^1 du \frac{\sqrt{1-u^{2(d-1)}}}{u^{d-1}\sqrt{1-u^d}} \right).$$

Therefore one arrives at

$$S_{\mathrm{BH}} \approx \frac{L^{d-2}}{4G_N}\left(\frac{1}{(d-2)\epsilon^{d-2}}+\frac{\ell}{2\rho_H^{d-1}}-\frac{c_2}{\rho_H^{d-2}}\right).$$

where c_2 is a positive number. For example for $d = 3, 4$ one gets $c_2 = 0.88, 0.33$, respectively.

Note that the first finite term in the above expression is proportional to the volume which is indeed the thermal entropy, while the second finite term is proportional to the area of the entangling region.

In general for a sphere with the radius R one gets

$$\begin{aligned} S &\sim c_1 \frac{R^{d-2}}{\epsilon^{d-2}} + c_2 \frac{R^{d-3}}{\epsilon^{d-3}} + \cdots \\ &+ c_d \ln \frac{R}{\epsilon}, \qquad\qquad \text{for} \quad d \text{ even} \\ &+ p_d \qquad\qquad\qquad \text{for} \quad d \text{ odd} \end{aligned}$$

c_d is a universal and is related with the conformal anomaly.

Time-dependent backgrounds

So far we have considered static case where we have a time slice on which we can define minimal surfaces. In the time-dependent case there is no a natural choice of the time-slices.

In Lorentzian geometry there is no minimal area surface. In order to resolve this issue we use the covariant holographic entanglement entropy which is

$$S_A(t) = \frac{\mathrm{Area}(\gamma_A(t))}{4G_N^{(d+2)}}$$

where $\gamma_A(t)$ is the extremal surface in the bulk Lorentzian spacetime with the boundary condition $\partial\gamma_A(t) = \partial A(t)$.

Strong subadditivity?

Example of time-dependent case: Black hole formation or Thermalization

$$\text{Geometry} \iff \text{State}$$

$$AdS \text{ solution} \iff \text{Vaccum state}$$
$$\text{Black hole} \iff \text{Excited state; thermal}$$

Let us perturbe a system so that the end point of the time evolution would be a thermal state. This might be done by a global quantum quench. Typically during evolution the system is out of equilibrium.

The thermalization process after a global quantum quench may be map to a black hole formation due to a gravitational collapse.

A quantum quench in the field theory may occurs due to a sudden change in the system which might be caused by turning on the source of an operator in an interval $\delta t \to 0$.

This change can excite the system to an excited state with non-zero energy density that could eventually thermalize to an equilibrium state.

From gravity point of view this might be described by a gravitational collapse of a thin shell of matter which can be modeled by an AdS-Vaidya metric.

$$dS^2 = \frac{R^2}{r^2}[f(r,v)dv^2 - 2drdv + d\vec{x}^2], \quad f(r,v) = 1 - m\theta(v)r^d$$

where r is the radial coordinate, x_is are spatial boundary coordinates and v is the null coordinate. Here $\theta(v)$ is the step function and therefore for $v < 0$ the geometry is an AdS metric while for $v > 0$ it is an AdS-Schwarzschild black hole.

This is a solution of Einstein gravity with matter. The energy momentum of the infalling matter is given by $T_{\mu\nu} = \varrho U_\mu U_\nu$ with $U_\mu = \delta_{\mu v}$, and

$$\varrho = \frac{1-d}{2}\frac{\partial f(r,v)}{\partial v}\, r,$$

Note that the null energy condition requires $\varrho > 0$.

Entanglement entropy for a strip

To compute the entanglement entropy for a strip with width ℓ, let us consider the following strip

$$-\frac{\ell}{2} \leq x_1 = x \leq \frac{\ell}{2}, \qquad 0 \leq x_a \leq L, \quad \text{for } a = 2, \cdots, D.$$

Since the metric is not static one needs to use the covariant proposal for the holographic entanglement entropy. Therefore the corresponding co-dimension two hypersurface in the bulk may be parametrized by $v(x)$ and $r(x)$. Then the induced metric on the hypersurface, setting $r = \rho$, is

$$ds^2_{\text{ind}} = \frac{1}{\rho^2}\left[\left(1 - f(\rho, v)v'^2 - 2v'\rho'\right)dx^2 + dx_a^2\right),$$

The area of the hypersurface reads

$$\mathcal{A} = \frac{L^{d-2}}{2} \int_{-\ell/2}^{\ell/2} dx \, \frac{\sqrt{1 - 2v'\rho' - v'^2 f}}{\rho^{d-1}}$$

We note, however, that since the action is independent of x the corresponding Hamiltonian is a constant of motion

$$\rho^n \mathcal{L} = H = \text{constant}.$$

Moreover we have two equations of motion for v and ρ. Indeed, by making use of the above conservation law the corresponding equations of motion read

$$\partial_x P_v = \frac{P_\rho^2}{2} \frac{\partial f}{\partial v}, \qquad \partial_x P_\rho = \frac{P_\rho^2}{2} \frac{\partial f}{\partial \rho} + \frac{n}{\rho^{2n+1}} H^2,$$

where

$$P_v = (\rho' + v' f), \qquad P_\rho = v',$$

are the momenta conjugate to v and ρ up to a factor of H^{-1}, respectively.

These equations have to be supplemented by the following boundary conditions

$$\rho(\frac{\ell}{2}) = 0, \qquad v(\frac{\ell}{2}) = t, \qquad \rho'(0) = 0, \qquad v'(0) = 0,$$

and

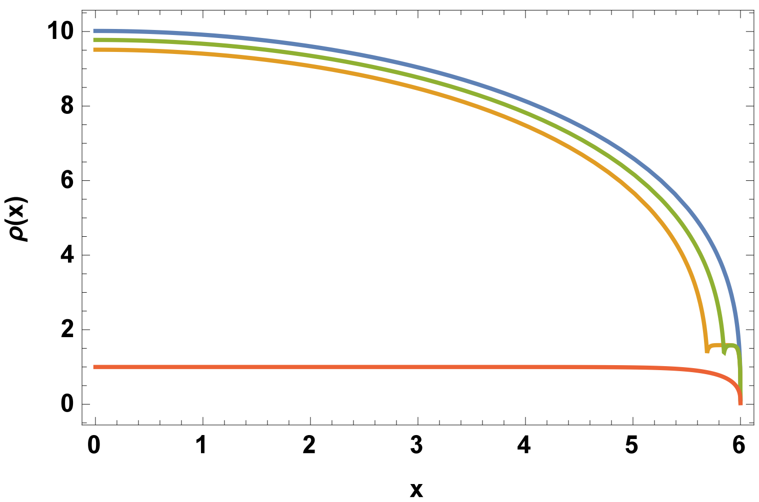
$$\rho(0) = \rho_t, \qquad v(0) = v_t,$$

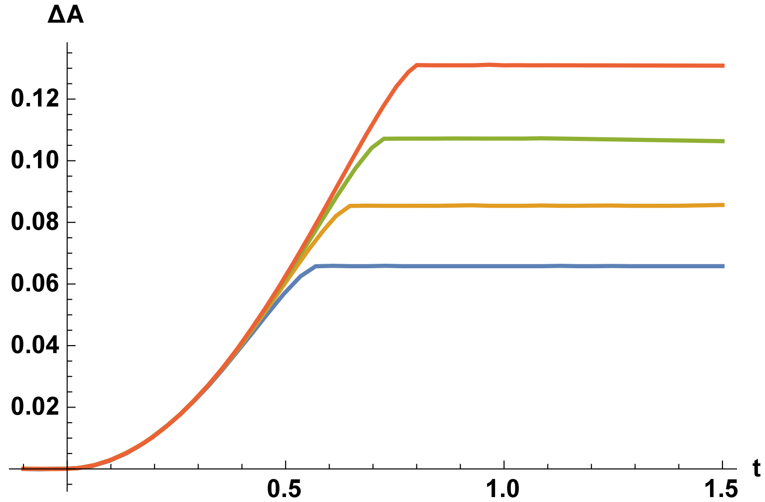
where (ρ_t, v_t) is the coordinate of the extremal hypersurface turning point in the bulk.

In what follows we will consider the case of $\ell \gg \rho_H$

Numerical results

$$m(v) = \frac{m_0}{2}(1 + \tanh \frac{v}{a})$$





The profile of the extremal surface for a strip with $\ell = 12$ at thin shell limit $a = 0.001$ with $\rho_H = 1$ for different boundary times: $t = 0, 5, 8, 10$.

Evolution of the regularized area of the minimal surface for $a = 0.001$ and $\rho_H = 1$. The small entangling regions for $\ell = 0.7, 0.8, 0.9, 1$.

Semi-Analytic

$$m(v) = m_0 \delta(v)$$

i) $v < 0$ **region**

In this case which corresponds to the vacuum solution one has

$$P_{(i)v} = \rho' + v' = 0,$$

which together with the conservation law yields to

$$v(\rho) = v_t + (\rho_t - \rho), \qquad x(\rho) = \int_{\rho}^{\rho_t} \frac{d\xi\ \xi^n}{\sqrt{\rho_t^{2n} - \xi^{2n}}}.$$

Note also that at the null shell where $v = 0$, from the above equation, one gets

$$\rho_c = \rho_t + v_t$$

which, indeed, gives the point where the extremal hypersurface intersects the null shell. Moreover, from the conservation law in the initial phase one finds

$$\rho'_{(i)} = -v'_{(i)} = -\sqrt{\left(\frac{\rho_t}{\rho_c}\right)^{2n} - 1}$$

ii) $v > 0$ region

In this case which the corresponding geometry is a the black hole, using the conservation law one arrives at

$$\rho'^2 = P^2_{(f)v} + \left(\left(\frac{\rho_t}{\rho}\right)^{2n} - 1\right) f(\rho) \equiv V_{eff}(\rho),$$

which can also be used to find

$$\frac{dv}{d\rho} = -\frac{1}{\tilde{f}(\rho)}\left(1 + \frac{P_{(f)v}}{\sqrt{V_{eff}(\rho)}}\right).$$

Here $V_{eff}(\rho)$ might be thought of as an effective potential for a one dimensional dynamical system whose dynamical variable is ρ. In particular the turning point of the potential can be found by setting $V_{eff}(\rho) = 0$.

iii) Matching at the null shell

Since ρ and v are the coordinates of the space time they should be continuous across the null shell.

We note. however, that since one is injecting matters along the null direction v, one would expect that its corresponding momentum conjugate jumps once one moves from $v < 0$ region to $v > 0$ region.

Therefore by integrating the equations of motion across the null shell one arrives at

$$\rho'_{(f)} = \left(1 - \frac{1}{2} g(\rho_c)\right) \rho'_{(i)}, \quad \mathcal{L}_{(f)} = \mathcal{L}_{(i)}, \quad v'_{(f)} = v'_{(i)}.$$

It is, then, easy to read the momentum conjugate of v in $v > 0$ region

$$P_{(f)v} = \frac{1}{2} g(\rho_c) \rho'_{(i)} = -\frac{1}{2} g(\rho_c) \sqrt{\left(\frac{\rho_t}{\rho_c}\right)^{2n} - 1}.$$

Now we have all ingredients to find the area of the corresponding extremal hypersurface in the bulk. In general the extremal hypersurface could extend in both $v < 0$ and $v > 0$ regions of space-time. Therefore the width ℓ and the boundary time are found

$$\frac{\ell}{2} = \int_{\rho_c}^{\rho_t} \frac{d\rho \, \rho^{d-1}}{\sqrt{\rho_t^{2(d-1)} - \rho^{2(d-1)}}} + \int_0^{\rho_c} \frac{d\rho}{\sqrt{V_{eff}(\rho)}}, \qquad t = \int_0^{\rho_c} \frac{d\rho}{f(\rho)} \left(1 + \frac{E}{\sqrt{V_{eff}(\rho)}}\right),$$

where $E = P_{(f)v}$.

Finally the entanglement reads

$$S = \frac{L^{d-2}}{2G} \left[\int_{\rho_c}^{\rho_t} \frac{\rho_t^{d-1} d\rho}{\rho^{d-1}\sqrt{\rho_t^{2(d-1)} - \rho^{2(d-1)}}} + \rho_t^{d-1} \int_0^{\rho_c} \frac{d\rho}{\rho^{2(d-1)}\sqrt{V_{eff}(\rho)}} \right].$$

Early time

At the early time where $t \ll \rho_H$ the crossing point of the hypersurfaces is very close to the boundary, $\frac{\rho_c}{\rho_H} \ll 1$. Therefore one may expand t, and A leading to

$$
\begin{aligned}
t &\approx \rho_c \left(1 + \frac{1}{d+1}\left(\frac{\rho_c}{\rho_H}\right)^d + \frac{1}{2d+1}\left(\frac{\rho_c}{\rho_H}\right)^{2d} + \ldots\right), \\
A &\approx \frac{L^{d-2}}{(d-2)}\left(\frac{1}{\epsilon^{d-2}} - c\frac{1}{\rho_t^{d-2}}\right) + \frac{L^{d-2}m}{4}\rho_c^2\left(1 + \frac{1}{2d}\left(\frac{\rho_c}{\rho_t}\right)^{2(d-1)} + \ldots\right),
\end{aligned}
$$

where $c = \sqrt{\pi}\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}$. So that at leading order one finds

$$S \approx S_{\text{vac}} + \frac{L^{D-1}m}{8G}t^2.$$

Intermediate time interval

In the intermediate time interval where $\rho_H \ll t \ll \frac{\ell}{2}$, the entanglement entropy growth linearly with time. Indeed, there is a critical extremal surface which is responsible for the linear growth in this time interval.

$V_{eff}(\rho)$ might be thought of as an effective potential for a one dimensional dynamical system whose dynamical variable is ρ.

For a fixed extremal hypersurface turning point in the bulk, ρ_t, there is a free parameter in the effective potential given by ρ_c which may be tuned to a particular value $\rho_c = \rho_c^*$ such that the minimum of the effective potential becomes zero

$$\left.\frac{\partial V_{eff}(\rho)}{\partial \rho}\right|_{\rho_m,\rho_c^*} = 0, \qquad\qquad V_{eff}(\rho)|_{\rho_m,\rho_c^*} = 0.$$

If the hypersurface intersects the null shell at the critical point it remains fixed at ρ_m.

Therefore in the intermediate time interval the main contributions to ℓ, t and A come from a hypersurface which is closed to the critical extremal hypersurface.

In this case assuming $\rho_c = \rho_c^*(1-\delta)$ for $\delta \ll 1$ in the limit of $\rho \to \rho_m$ and with the conditions $\frac{\rho_c^*}{\rho_t}, \frac{\rho_m}{\rho_t} \ll 1$ one finds

$$
\begin{aligned}
t &\approx -\frac{\rho_m^{2(z-1)} E^*}{f(\rho_m)\sqrt{\frac{1}{2}V''_{eff}}} \log \delta, \qquad \frac{\ell}{2} \approx c\rho_t + \frac{f(\rho_m)}{E^*} t \\
A &\approx \frac{L^{d-2}}{(d-2)}\left(\frac{1}{\epsilon^{d-2}} - c\frac{1}{\rho_t^{d-2}}\right) - \frac{L^{d-2}\rho_t^{d-1}}{\rho_m^{2(d-1)}\sqrt{\frac{1}{2}V''_{eff}}} \log \delta
\end{aligned}
$$

where $E^* \equiv E(\rho_c^*)$. So that

$$S \approx S_{\text{vac}} + L^{d-2} S_{\text{th}}\, v_E\, t.$$

The scaling behaviors of entanglement entropy

- Early times growth where $t \ll \rho_H$

$$\Delta S \approx \frac{L^{d-2} m}{8G} t^2,$$

- The intermediate region where $\frac{\ell}{2} \gg t \gg \rho_H$

$$\Delta S \approx L^{d-2} S_{\text{th}}\, v_E\, t,$$

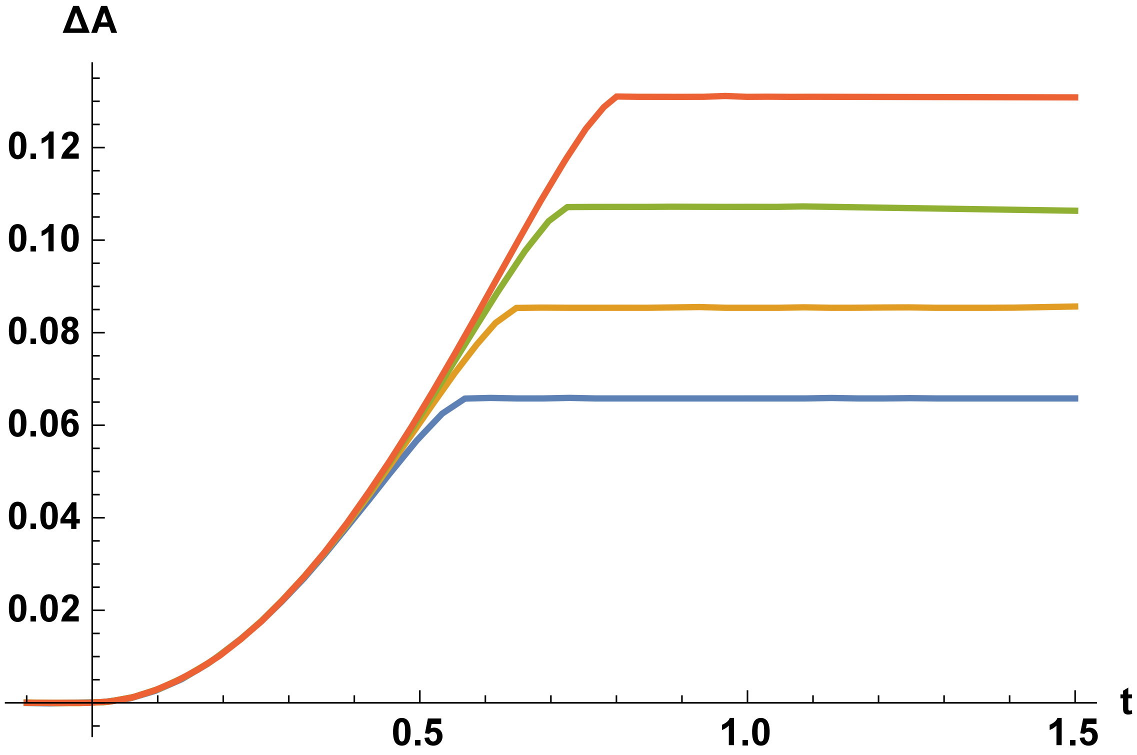
where

$$v_E = \left(\frac{d-2}{2(d-1)}\right)^{\frac{d-1}{d}} \sqrt{\frac{d}{d-2}}, \qquad S_{\text{th}} = \frac{1}{4G\rho_H^{d-1}}$$

- Late time saturation $t \sim \frac{\ell}{2}$

$$\Delta S \approx \frac{L^{d-2}\ell}{4G\rho_H^{d-1}} = L^{d-2}\ell S_{\text{th}}\ .$$

Recall



Having reviewed the computation of holographic entanglement entropy for a static and time dependent background Let us consider

Special topics on holographic entanglement entropy

1. Entanglement thermodynamics

Using our holographic model we have explored time dependent behaviors of holographic entanglement entropy. The observation may be summarized as follows.

The system has to scales: the size of entangling region ℓ and the radius of horizon ρ_H.

Therefore we have two time scales

$$t \sim \rho_H \qquad \text{local equilibrium,}$$

$$t \sim \frac{\ell}{2} \qquad \text{saturation on entanglement entropy.}$$

When $\rho_H > \frac{\ell}{2}$ the entanglement entropy saturates at $t \sim \frac{\ell}{2}$ before the system reaches a local equilibrium, whereas for $\rho_H < \frac{\ell}{2}$ the entanglement entropy is far from its equilibrium value even though the system is locally equilibrated.

For $\frac{\ell}{2} < \rho_H$ one has

$$\text{Early times} \qquad S \sim S_{\text{vac}} + V_{d-1}\,\mathcal{E}\,t^2,$$

$$\text{Saturation} \qquad S \sim S_{\text{vac}} + V_{d-1}\,\mathcal{E}\,\frac{\ell^2}{4}.$$

Here $\mathcal{E}$ is the energy density. Since the system has not reached a local equilibrium, this is the quantity one may define.

For $\frac{\ell}{2} < \rho_H$ one has

$$\begin{aligned} &\text{Early times} && S \sim S_{\text{vac}} + V_{d-1}\,\mathcal{E}\,t^2, \\ &\text{Intermediate} && S \sim S_{\text{vac}} + V_{d-1}\,S_{th}\,t, \\ &\text{Saturation} && S \sim S_{\text{vac}} + V_{d-1}\,S_{th}\,\frac{\ell}{2} + \frac{V_{d-1}}{\rho_H^{d-2}}. \end{aligned}$$

The intermediate region is $\rho_H < t < \frac{\ell}{2}$. So that at the early times the system is out of equilibrium, though the system reaches a local equilibrium while the entanglement entropy still grows with time.

After the local equilibrium the entanglement entropy may be given in terms of the thermal entropy.

The entanglement entropy at the early times is sensitive to the state, while in the intermediate region it always grows linearly.

How general these behaviors are?Could entanglement entropy provide a general framework to study a system out of equilibrium?

Let me remained you that thermodynamics provides a useful tool to study a system when it is in the thermal equilibrium. In this limit the physics may be described in terms of few macroscopic quantities such as energy, temperature, pressure, entropy.

There are also laws of thermodynamics which describe how these quantities behave under various conditions. In particular the first law of thermodynamics which is energy conservation, tells us how the entropy change as one changes the energy of the system.

There are several interesting phenomena which occur when the system is far from thermal equilibrium.

The entanglement entropy may provide a useful quantity to study excited quantum systems which are far from thermal equilibrium. For a generic quantum system it is difficult to compute the entanglement entropy. Nevertheless, at least, for those quantum systems which have holographic descriptions, one may use the holographic entanglement entropy to explore the behavior of the system.

Another quantity which can be always defined is the energy (or energy density) of the system. It is then natural to pose the question whether there is a relation between the entanglement entropy of an excited state and its energy.

For sufficiently small subsystem, the entanglement entropy is proportional to the energy of the subsystem. The proportionality constant is indeed given by the size of the entangling region.

Recall that in a black hole geometry and in the limit of $m\ell^d \ll 1$ one has

$$S_{\rm BH} = S_{\rm vac} + \frac{L^{d-2}}{4G_N} c_1 m\ell^2.$$

On the other hand the energy of the black hole is

$$E = \frac{dL^{d-2}\ell}{16\pi G_N} m$$

Therefore one arrives at

$$E = T_E \Delta S, \qquad\qquad T_E \sim \frac{1}{\ell}$$

This may be considered as The first law of entanglement thermodynamics.

It may also be recast into the following form

$$\Delta S = \Delta H.$$

Modular Hamiltonian

Consider a general quantum system. For any state in the system, the state of a subsystem A is described by reduced density matrix

$$\rho_A = \mathrm{Tr}_{A_c}(\rho_{\mathrm{total}})$$

where ρ_{total} is the density matrix of the system and A_c is the complement of A.

The entanglement entropy is defined by the von Neumann entropy

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A$$

Since the reduced density matrix is both Hermitian and positive (semi) definite, it may be expressed as

$$\rho_A = \frac{e^{-H_A}}{\mathrm{Tr}(e^{-H_A})}, \qquad \mathrm{Tr}(\rho_A) = 1$$

H_A is modular Hamiltonian.

Consider any infinitesimal variation to the state of the system. At first order one gets

$$\begin{aligned} \delta S_A &= -\mathrm{Tr}(\delta\rho_A \log \rho_A) - \mathrm{Tr}(\rho_A \rho_A^{-1} \delta\rho_A) \\ &= \mathrm{Tr}(\delta\rho_A H_A) - \mathrm{Tr}(\delta\rho_A) \end{aligned}$$

Therefore the variation of entanglement entropy satisfies

$$\delta S_A = \delta\langle H_A \rangle \qquad \text{First law}$$

where H_A is associated with the original unperturbed state.

For a thermal state $\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$ one gets $T\delta S = \delta\langle H \rangle$.

The strong subadditivity may be thought of as the second law.

For a general quantum field theory, general state and general entangling region, the modular Hamiltonian it not known.

For a conformal field theory in its vacuum state $\rho_{\text{total}} = |0\rangle\langle 0|$ in d-dimensional Minkowski space and an entangling region in a form of a ball, the modular Hamiltonian has a simple form.

Consider a ball with radius R_0 on a time slice $t = t_0$ and centered at $x^i = x_0^i$ one has

$$H_{\text{Ball}} = 2\pi \int_{\text{Ball}} d^{d-1}x \, \frac{R_0^2 - |\vec{x} - \vec{x}_0|^2}{2R_0} T_{tt}(t_0, \vec{x})$$

where $T_{\mu\nu}$ is stress tensor.

Therefore, starting from the vacuum state for any CFT and a ball-shaped entangling region, the first law reduces to

$$\delta S_B = \delta E_B$$

where

$$E_B = 2\pi \int_{\mathrm{Ball}} d^{d-1}x \; \frac{R_0^2 - |\vec{x} - \vec{x}_0|^2}{2R_0} \langle T_{tt}(t_0, \vec{x}) \rangle$$

For a CFT theory which has gravitational description, both δS and δE can be computed from gravity side. Then the first law is a constraint on the small perturbations around the vacuum AdS solution.

- For small perturbation around the vacuum solution satisfies linear equations of motion, the first law would hold.

- A small perturbation which satisfies first law, will obey linear equations of motion.

Application

First law applied to infinitesimal ball shaped entangling regions may be used to compute holographic stress tensor and constrains the asymptotic behavior of the metric.

Given an event in the dual field theory one would like to know which part of bulk could describe this event?

2. n-partite information

One may study entanglement entropy for two disjoint regions. For two disjoint regions A and B, it is more natural to compute the amount of correlations (both classical and quantum) between these two regions which is given by the mutual information.

It is actually a quantity which measures the amount of information that A and B can share which in terms of the entanglement entropy is given by

$$I(A,B) = S(A) + S(B) - S(A \cup B),$$

Although the entanglement entropy is UV divergent, the mutual information is finite. Moreover by making use of the subadditivity property of the entanglement entropy, it is evident that the mutual information is always non-negative and it is zero for two uncorrelated systems.

More generally one may want to compute entanglement entropy for a subsystem consists of n disjoint regions $A_i,\ i = 1, \cdots, n$.

Following the notion of mutual information for a system of two disjoint regions, it is natural to define a quantity, n-partite information, which could measure the amount of information or correlations (both classical and quantum) between them. Intuitively, one would expect that for n un-correlated systems the n-partite information must be zero. Moreover, for n disconnected systems it should be finite.

Actually for a given n disjoint regions, there is no a unique way to define n-partite information and indeed, it can be defined in different ways. In particular in terms of entanglement entropy one may define the n-partite information as follows

$$\begin{aligned} I^{[n]}(A_{\{i\}}) \;=\; & \sum_{i=1}^{n} S(A_i) - \sum_{i<j}^{n} S(A_i \cup A_j) + \sum_{i<j<k}^{n} S(A_i \cup A_j \cup A_k) - \cdots\cdots \\ & -(-1)^n S(A_1 \cup A_2 \cup \cdots \cup A_n), \end{aligned}$$

In terms of the mutual information, this n-partite information may be recast into the following form

$$\begin{aligned} I^{[n]}(A_{\{i\}}) \;=\; & \sum_{i=2}^{n} I^{[2]}(A_1, A_i) - \sum_{i=2<j}^{n} I^{[2]}(A_1, A_i \cup A_j) \\ & + \sum_{i=2<j<k}^{n} I^{[2]}(A_1, A_i \cup A_j \cup A_k) - \cdots \\ & +(-1)^n I^{[2]}(A_1, A_2 \cup A_2 \cdots \cup A_n). \end{aligned}$$

It is worth mentioning that although the mutual information is always non-negative, the n-partite information $I^{[n]}$ could have either signs.

It may be re-expressed in terms of $(n-1)$-partite information as follows

$$
\begin{aligned}
I^{[n]}(A_{\{i\}}) \;=\;& I^{[n-1]}(A_{\{1,\cdots,n-2\}}, A_{n-1}) + I^{[n-1]}(A_{\{1,\cdots,n-2\}}, A_n) \\
& - I^{[n-1]}(A_{\{1,\cdots,n-2\}}, A_{n-1} \cup A_n).
\end{aligned}
$$

n-partite information $I^{[n]}$ may be thought of a quantity which measures the degree of extensivity of the $(n-1)$-partite information.

In the literature of information theory for a subsystem consisting of n disjoint regions, one may define another quantity which, indeed, is a direct generalization of mutual information known as multi-partite entanglement defined as follows

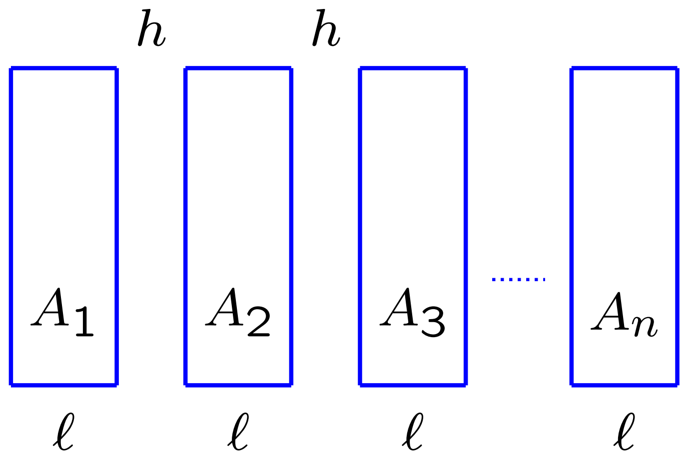
$$J^{[n]}(A_{\{i\}}) = \sum_{i}^{n} S(A_i) - S(A_1 \cup A_2 \cup \cdots \cup A_n),$$

In terms of the mutual information it may be recast into the following form

$$J^{[n]}(A_{\{i\}}) = I^{[2]}(A_1, A_2) + I^{[2]}(A_1 \cup A_2, A_3) + \cdots + I^{[2]}(A_1 \cup A_2 \cdots \cup A_{n-1}, A_n).$$

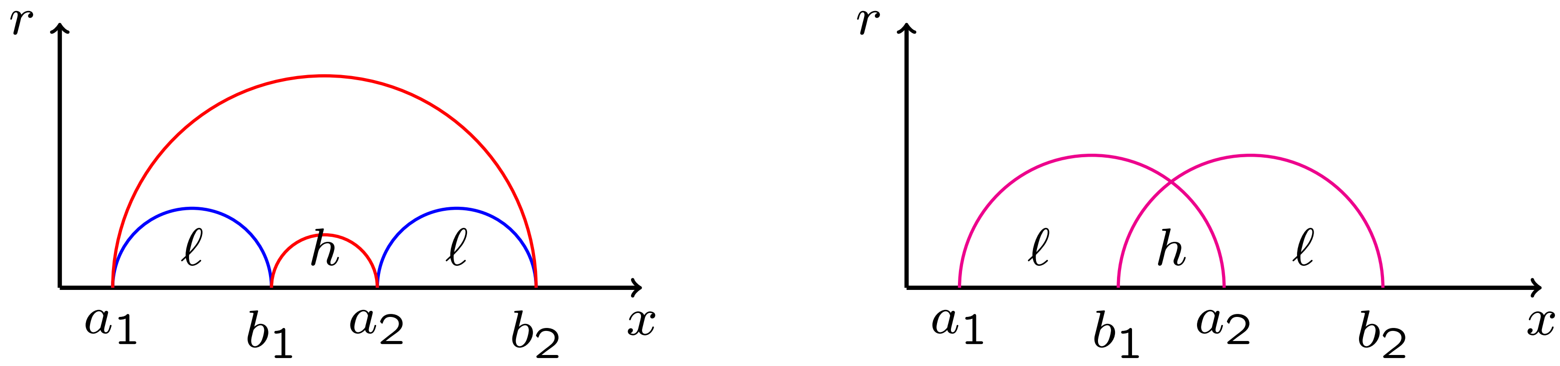
Note that this quantity is finite for a system with n disjoint regions and is zero for n un-correlated regions. It is always non-negative.

Holographic n-partite information



We will study n-partite information of a subsystem consists of n disjoint regions A_i, $i = 1, \cdots, n$ in a d-dimensional CFT for the vacuum and thermal states whose gravity duals are provided by the AdS and AdS black brane geometries. The n disjoint regions are given by n parallel infinite strips of equal width ℓ separated by $n-1$ regions of width h.

$$
\begin{aligned}
I^{[n]}(A_{\{i\}}) &= \sum_{i=1}^{n} S(A_i) - \sum_{i<j}^{n} S(A_i \cup A_j) + \sum_{i<j<k}^{n} S(A_i \cup A_j \cup A_k) - \cdots\cdots \\
&- (-1)^n S(A_1 \cup A_2 \cup \cdots \cup A_n).
\end{aligned}
$$



The main subtlety in evaluating the above quantity is the computation of entanglement entropy of union of subsystem.

For a given two strips with the widths ℓ and distance h, there are two minimal hypersurfaces associated with the entanglement entropy $S(A \cup B)$ and thus the corresponding entanglement entropy behaves differently.

$$S(A \cup B) = \begin{cases} S(2\ell + h) + S(h) & h \ll \ell, \\ 2S(\ell) & h \gg \ell, \end{cases}$$

Therefore the mutual information becomes

$$I(A \cup B) = \begin{cases} 2S(\ell) - S(2\ell + h) - S(h) & h \ll \ell, \\ \\ 0 & h \gg \ell, \end{cases}$$

The holographic mutual information undergoes a first order phase transition as one increases the distance between two strips. Indeed, there is a critical value of $\frac{h}{\ell}$ above which the mutual information vanishes. As we just observed, this peculiar behavior has to do with the definition of entanglement entropy of the union $A \cup B$.

n-partite information for the vacuum state of a CFT whose gravity dual is given by an AdS background

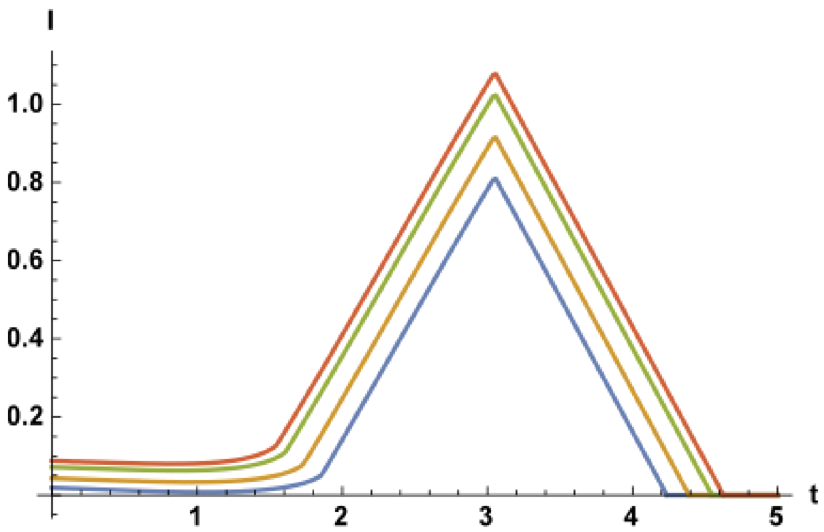
$$\tilde{I}^{[n]}_{\rm vac} = \frac{L^{d-2}c_0}{4G_N}\Big(-\frac{2}{((n-1)\ell+(n-2)h)^{d-2}}+\frac{1}{(n\ell+(n-1)h)^{d-2}} + \frac{1}{((n-2)\ell+(n-3)h)^{d-2}}\Big).$$

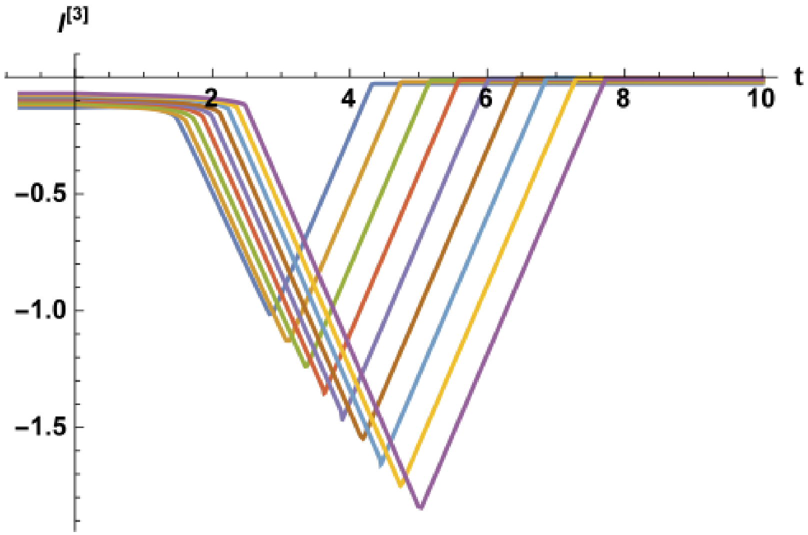
For a thermal state whose gravity dual is provided by an AdS black brane geometry, and in the limit of $\ell \ll \rho_H$, one finds

$$\tilde{I}^{[n]}_{\rm BH} = \tilde{I}^{[n]}_{\rm vac} - \frac{L^{d-2}}{2G_N}c_1\frac{(\ell+h)^2}{\rho_H^d}.$$

while for $\rho_H \ll \ell$ it vanishes.

time dependent behavior





Within the context of the AdS/CFT correspondence one may compute n-partite information. It has a definite sign: for even n it is positive and for odd n it is negative, though for a generic field theory it could have either signs.

One may suspect that having definite sign for the n-partite information is, indeed, an intrinsic property of a field theory which has gravity dual.

What about other shape? Can we see the phase transition for n-partite using the field theory description?

3. Area law?

We have seen that the entanglement entropy is proportional to the area of the entangling region. How general is this?

Already in two dimensions the entanglement entropy is proportional to the log

$$S = \frac{c}{3} \ln \frac{\ell}{\epsilon}$$

It is possible to have other behavior. In particular for the case where the corresponding theory is non-local.

Let us explore it in an explicit example.

General solution with hyperscaling factor

$$S = -\frac{1}{16\pi G_N}\int d^{D+2}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{N_g} e^{\lambda_i\phi}F^{(i)2}\right],$$

where $V(\phi) = V_0 e^{\gamma\phi}$, G is the Newton constant, γ, V_0 and λ_i are free parameters of the model.

One of the gauge field is required to produce an anisotropy while the above particular form of the potential is needed to get hyperscaling violating factor. The other gauge fields make the background charged. In what follows we will consider $N_g = 2$.

The model admits solutions with hyperscaling violating factor

$$ds^2 = r^{-2\frac{\theta}{D}}\left(-r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2\right),$$

Under scaling

$$t \to \xi^z t, \qquad x_i \to \xi x, \qquad r \to \xi^{-1} r$$

the metric scales $ds \to \xi^{\theta/D} ds$.

$$S \sim T^{(D-\theta)/z}$$

It has exact charged black hole solutions as follows

$$ds^2 = r^{-2\frac{\theta}{D}}\left(-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2\right), \quad \phi = \beta \ln r,$$

$$A_t^{(1)} = \sqrt{\frac{2(z-1)}{D-\theta+z}}\, r^{D-\theta+z}, \qquad A_t^{(2)} = \sqrt{\frac{2(D-\theta)}{D-\theta+z-2}}\, \frac{Q}{r^{D-\theta+z-2}},$$

with $\beta = \sqrt{2(D-\theta)(z-1-\theta/D)}$ and

$$f(r) = 1 - \frac{m}{r^{D-\theta+z}} + \frac{Q^2}{r^{2(D-\theta+z-1)}}.$$

where z is the dynamical exponent and θ is the hyperscaling violation exponent.

To be more concrete, consider $m = Q = 0$ and after a double Wick rotation as follows

$$t \to iy, \qquad x_d \to it,$$

one gets

$$ds^2_{d+2} = r^{\frac{2\theta}{d}} \left(\frac{dy^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{\sum_{i=1}^{d-1} dx_i^2}{r^2} - \frac{dt^2}{r^2} \right).$$

Let us compute the holographic entanglement entropy for the following strip

$$\frac{\ell}{2} \le y \le \frac{\ell}{2}, \qquad 0 \le x_i \le L, \ \text{for } i = 1, \cdots, d-1.$$

Setting $y = y(r)$ the induced metric of the co-dimension two hyper surface is

$$ds^2_{ind} = r^{2\frac{\theta}{d}} \left[\left(\frac{y'^2}{r^{2z}} + \frac{1}{r^2} \right) dr^2 + \frac{\sum_{i=1}^{d-1} dx_i^2}{r^2} \right].$$

Therefore the area of the surface is

$$A = L^{d-1} \int_\epsilon dr \, \frac{\sqrt{r^{2(z-1)} + y'^2}}{r^{d+z-\theta-1}}.$$

Minimizing this area, for general θ, d and z, one finds

$$S = \frac{L^{d-1}}{4(d-\theta-1)G_N} \left(\frac{1}{\epsilon^{d-\theta-1}} - b_0 \frac{c_0^{(d-\theta-1)/z}}{\ell^{(d-\theta-1)/z}} \right).$$

For $\theta = d - 1$

$$S = \frac{1}{4z\pi G_N} \frac{L^{d-1}}{r_F^{d-1}} \ln \frac{z\ell}{\epsilon^z},$$

For $\theta = d$

$$S \sim L^{d-1} \ell^{1/z}.$$

For $z = 1$ it is indeed a volume law!

The properties of the system may be reflected in the behavior of the holographic Entanglement. May be used as probe: Different phase transitions, Fermi surface,

4. Higher derivative

The holographic formula we have considered is for Einstein gravity. Motivated by the Wald formula it is interesting to see how this formula is modified in the presence of higher derivative corrections to Einstein gravity.

Unlike the Wald formula for black hole entropy there is no a rigorous derivation for a general expression when we have arbitrary higher derivative corrections.

Consider an action with R^2 terms

$$S = -\frac{1}{16G_N}\int d^{d+2}x\sqrt{g}\left[R - 2\Lambda + (\alpha R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^2)\right],$$

For the Gauss-Bonnet gravity where $\alpha = \lambda, \beta = -4\lambda, \gamma = \lambda$, the holographic entanglement entropy is argued to be

$$S_A = \text{Min}_{\gamma_A}\left[\frac{1}{4G_N}\int_{\gamma_A} d^d x\sqrt{h}(1 + 2\lambda R_{int})\right],$$

where R_{int} is the intrinsic curvature of γ_A.

Nevertheless for generic case one still needs a general formula!

The main problem comes from the fact that, unlike horizon, for a generic hypersurface the extrinsic curvature is non-zero.

Therefore beyond the terms as that of Wald formula one could have other terms with is proportional to extrinsic curvature.

The corresponding entropy functional for our case becomes

$$S_A \sim \int d^2\zeta \sqrt{h}\left[\gamma R - \beta\left(R_{\mu\nu}n_i^\mu n_i^\nu - \frac{1}{2}\mathcal{K}^i\mathcal{K}_i\right) + \alpha\left(R_{\mu\nu\rho\sigma}n_i^\mu n_j^\nu n_i^\rho n_j^\sigma - \mathcal{K}^i_{\mu\nu}\mathcal{K}_i^{\mu\nu}\right)\right]$$

where $i = 1,2$ denotes two transverse directions to a co-dimension two hyper-surface in the bulk, n_i^μ are two unit mutually orthogonal normal vectors on the co-dimension two hyper-surface and $\mathcal{K}^{(i)}$ is the trace of two extrinsic curvature tensors defined by

$$\mathcal{K}^{(i)}_{\mu\nu} = \pi^\sigma{}_\mu \pi^\rho{}_\nu \nabla_\rho (n_i)_\sigma, \qquad \text{with} \quad \pi^\sigma{}_\mu = \epsilon^\sigma{}_\mu + \xi \sum_{i=1,2}(n_i)^\sigma (n_i)_\mu$$

where $\xi = -1$ for space-like and $\xi = 1$ for time-like vectors. Moreover h is the induced metric on the hyper-surface whose coordinates are denoted by ζ.

A way to find a reasonable expression is to use the replica trick which in general leads to a singular geometry. Then one should extract the contribution of the cone!

Near the cone the metric may be written as

$$ds^2 = g(r)d\tau^2 + dr^2 + \gamma_{ij}(r,x)dx^i dx^j \qquad g(r) \sim r^2 + \mathcal{O}(r^4)$$

with the identification $\tau \equiv \tau + 2\pi n$.

One may regularized the cone

$$ds^2 = e^{2\sigma(x,r)}[d\tau^2 + f_n(r)dr^2 + \gamma_{ij}(r,\tau,x)dx^i dx^j, \qquad f_n(r) = \frac{r^2 + b^2 n^2}{r^2 + b^2}$$

$$\gamma(r,\tau,x) = h_{ij}(x) + 2K^a_{ij} n^a r^n + g_{ij}(x) r^2 + (K^a K^b)_{ij} n^a n^b r^{2n} + \cdots$$

Other regularizations may be used: The results should be independent of the regularization.

Using this metric one can find the contribution of each rearm . For example

$$R^{(n)}_{\mu\nu} = R^{\text{reg}}_{\mu\nu} + 2\pi n^a_\mu n^b_\nu \delta_\Sigma$$

which leads to a term in the entropy as follows

$$S_A \to \int_\Sigma R_{\mu\nu} n^\mu_a n^\nu_b$$

On the other hand from the extrinsic curvature one gets

$$S_A \to -\frac{1}{2}\int_\Sigma K^2$$

So one arrives that

$$\int R_{\mu\nu} R^{\mu\nu} \to \int_\Sigma (R_{\mu\nu} n^\mu_a n^\nu_b - \frac{1}{2} K^2)$$

Other terms may be computed in the same way.

Is this the right thing to do? What about the regularization?

Let us check it for 4D conformal gravity

$$
\begin{aligned}
S &= -\frac{\kappa}{32\pi}\int d^4x\,\sqrt{-g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) \\
&= -\frac{\kappa}{32\pi}\mathrm{GB}_4 - \frac{\kappa}{16\pi}\int d^4x\,\sqrt{-g}\,\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right),
\end{aligned}
$$

where GB_4 is the four dimensional Gauss-Bonnet action which is a total derivative and does not contribute to the equations of motion. Note that since the Gauss-Bonnet term is topological, the whole dynamics must be encoded in the second term.

It is then easy to compute the entanglement entropy (for example for and AdS solution)

$$S_{EE}^{\mathsf{dyn}} = \kappa L_y \left[\frac{1}{\epsilon} - \frac{2\pi\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} \frac{1}{\ell} \right].$$

Going through the same procedure for the Gauss-Bonnet term, one arrives at

$$S_{EE}^{\mathsf{GB}} = \kappa L_y \left(-\frac{1}{\epsilon} \right).$$

It is then clear that taking both contributions into account the divergent term will drop leading to a finite entanglement entropy.

More over S_{EE}^{dyn} is the same as that of Einstein gravity.