Scale *v.* Conformal Invariance for Carrollian Symmetry

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Statement

Scale invariance plus Poincare invariance in a Physical theory, leads to conformal invariance if the theory is:

I. Unitary
II. Has conserved energy-momentum tensor
III. Local

Energy-Momentum Tensor

Conserved Energy-Momentum Tensor:

$$\partial_\alpha T^{\alpha\beta} = 0$$

then:

$$\partial_\alpha (x_\beta \, T^{\alpha\beta}) = 0$$

If the trace vanishes we have a conserved current giving the dilations (scale transformations) :

$$D^\beta = x_\beta \, T^{\alpha\beta}$$

Zamalodchikov-Polchinski Theorem d=2

Suppose there exists φ such that

$$T^{\alpha\beta} = \partial^{\alpha}\partial^{\beta}\varphi$$

(the energy-momentum tensor can always be made symmetric Belifante 1940)

Then traceless condition says:

$$T = \partial^{2}\varphi = 0$$

i.e. φ is harmonic, easily done in 2d by choosing holomorphic functions.

D=4 ?

In higher dimensions this theorem may hold
Not easy to give a proof , still ambiguous .

Nakayama, Yu. "Scale invariance vs conformal invariance." *Physics Reports* (2014).

"…there is no known example of scale invariant but non-conformal field theories in d = 4 under the assumptions of unitarity and …
We have a perturbative proof based on the higher dimensional analogue of Zamolodchikov's c-theorem, but the non-perturbative proof is yet to come"

D=4 ?

Kara Farnsworth, Markus A. Luty, Valentina Prelipina (2013) :

Non-perturbative proof

Adam Bzowski, Kostas Skenderis (2014):

Not True !! High momentum does not always correspond to short distance (OPE)

Anatoly Dymarsky, Kara Farnsworth, Zohar Komargodski, Markus A. Luty, Valentina Prilepina (2014)

If energy-momentum tensor is not generalized free field (product of 2-point functions) then CFT

Question:

Does the same property hold for Field Theories over other space times ?

It might do, for space-times you get by contraction of Minkowsky space-time

Scale vs Conformal Symmetry outside of Lorentz invariance

Contraction of the Conformal algebra to the non-relativistic limit leaves us with the Conformal Galilean Algebra (CGA)

So we may ask: does scale invariance + other Space-time symmetries such as Galilean Symmetry lead to conformal symmetry ?

Scale vs Conformal Symmetry outside of Lorentz invariance

Attempt to answer this question by contraction:
<u>The non-relativistic $c \to \infty$ limit does not work</u>

The other limit i.e. The ultra relativistic limit can be taken by letting the speed of light tend to zero, and we expect this to produce the Levy-Leblond symmetry

J. M. L´evy-Leblond, "Une nouvelle limite non-relativiste du group de Poincar´e," Ann. Inst. H. Poincar´e, 3 (1965) 1.
2-Duval Gibbons, Horvathy "Conformal Carroll groups" Arxiv 1403.4213

Carroll group

The symmetry was named Carroll Group by Levy-Leblond. A mathematical curiosity.

- Massive particles are motionless, so all dynamics is “photons”
- May be relevant to Tachyons (Gibbons)
- In d=1+1 is isomorphic to Conformal Galilean Algebra (CGA)

1-Bagchi, et al. "Holography of 3D flat cosmological horizons." *Physical review letters* 110.14 (2013): 141302
2-Duval Gibbons, Horvathy “Conformal Carroll groups” Arxiv 1403.4213

Carroll group

The appearance of CGA in 1+1 dimensions[1] is an interesting coincidence , which is due to Carroll and Galilei algebras being isomorphic in 1+1 dimensions.

This may be understood as a kind of duality between $c \to 0$ and $c \to \infty$

1-Bagchi, et al. "Holography of 3D flat cosmological horizons." *Physical review letters* 110.14 (2013): 141302
2-Duval Gibbons, Horvathy “Conformal Carroll groups” Arxiv 1403.4213

Conservation laws in 2d

In 2d we have two simple conservation laws:

$$\partial_t h - c\,\partial_x p = 0$$

$$\partial_t p + \partial_x T = 0$$

Where h is the energy, *p* is the momentum density and *T* is the stress density.

Conservation laws in 2d

Under contraction c$\rightarrow$ 0 , This leads to :

$$\partial_t h = 0$$

$$\partial_t p + \partial_x T = 0$$

Dilations

dilations is now given by the density:

$$D = th + xp$$

Conservation of dilations implies that:

$$\partial_t D = h + T - \partial_x(xT)$$

Which is a conservation law if we have :

$$h + T = 0$$

This is exactly the traceless condition

Other symmetries

Boost: $B = -x\, h$

Acceleration: $F = x^2 h$

Special conformal transformation :

$$G = x^2 p + 2\, x^2 h$$

conservation law:

$$\partial_t G = -\partial_x (x^2 T)$$

Carrollian algebra

Straightforward to check that the Symmetry Algebra is satisfied:

$[E; P] = 0; \ [E;D] = E; \ [E;B] = 0; \quad [E; F] = 0; \quad [E;G] = -2B;$

$[P;D] = P; \ [P;B] =- E; \ [P; F] =- 2B; \ [P;G] = 2D; \quad [D;B] = 0;$

$[D; F] = F; \ [D;G] = G; \ [B; F] = 0; \quad [B;G] =- F; \quad [F;G] = 0;$

This algebra has an affine extension which includes “super translations” and “super rotations” , as found by BMS

Bagchi, Arjun, and Reza Fareghbal. "BMS/GCA redux: towards flats pace holography from non-relativistic symmetries." Journal of High Energy Physics 2012.10 (2012): 1-30.
Barnich, G., & Compere, G. (2006). Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions. *arXiv preprint gr-qc/0610130*.

Can we get the Carroll group out of flat space holography ?

We observe that the above symmetry is exactly the asymptotic symmetry of asymptotically flat space.

This was originally observed for 4-manifolds –BMS 1962

However also true in 3d – Ashtekar 1998

There is a proposal that the holographic dual of asymptotically flat space-times are conformal field theories with Carroll symmetry – 2012

Bondi Metzner Sachs (1962), Ashtekar et al (1998),
Bagchi, Arjun, and Reza Fareghbal. (2012).

Introduction to AdS/CFT correspondence

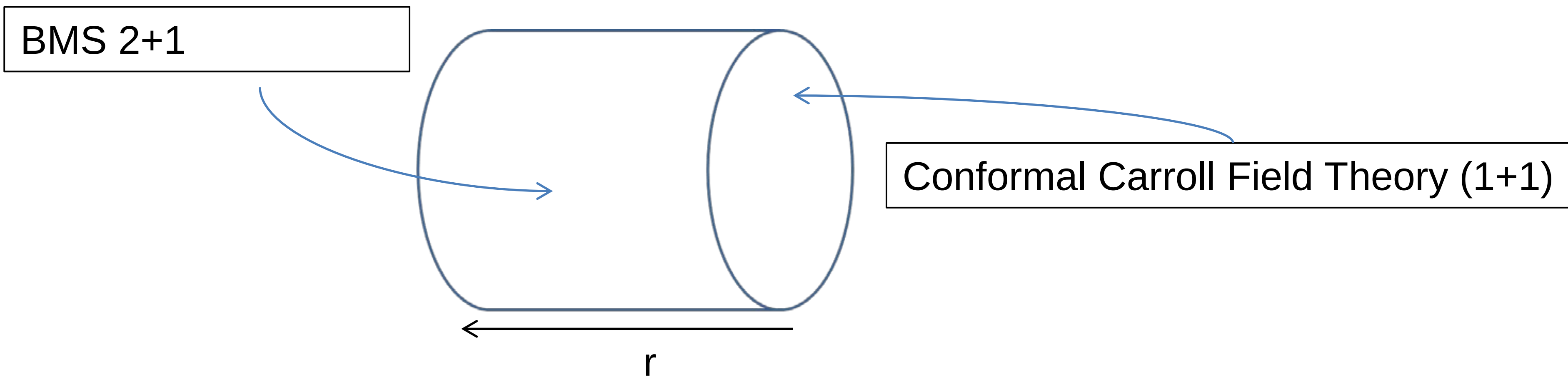
The AdS/CFT correspondence relates "string theory" on (locally asymptotically) AdS spacetimes with a
Conformal field theory (CFT) "residing" on the conformal boundary of the bulk spacetime.

In the usual application "string Theory" is taken at the semi-classical approximation, So we have a classical field φ theory on an asymptotically AdS back ground; g :

$$e^{-S_{eff}[g,\varphi]} \sim \left\langle e^{\int j\phi} \right\rangle$$

Here j is the asymptotic value of the classical field φ which acts as source for the quantum field ϕ , residing at
the boundary.

Flat Space Holography



$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right) c^2 dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\varphi^2$$

Bagchi, A., Detournay, S., Fareghbal, R., & Simón, J. (2013) *Physical review letters*, *110*(14), 141302.

Flat Space Holography

Bondi-Metzner-Sachs asked for the set of diffeomorphisms that preserve the asymptotic flatness condition.

This is interesting for study of scattering when you think of “vacuum” at infinity. Andrew Strominger, Alexander Zhiboedov (2014)

We therefore look at the flat limit of AdS , so let the AdS radius tend to infinity

$$L \to \infty$$

Flat Space Holography

The boundary metric is then given by :

$$ds^2{}_{boundary} = \frac{r^2}{G^2}\left(-\frac{G^2c^2}{L^2}dt^2 + G^2 d\varphi^2\right) + o\left(\frac{1}{r}\right)$$

Thus the induced speed of light on the boundary is:

$$\bar{c} = \frac{G\ c}{L}$$

Hence in the flat space limit :

$$L \to \infty$$

The speed of light tends to zero and we end up with Carroll Group

$$\bar{c} \to 0$$

Flat Holography and emergence of CGA

On the bulk side this give the Bondi-Metzner-Sachs (BMS) algebra.

On the boundary this is a contraction on the asymptotic Virasoro algebra giving centrally extended CGA, with central charges: $c_L = 0, c_M = \frac{1}{4}$.

This corresponds to the Carrollian limit ($c \to 0$) .

A. Bagchi, Phys.Rev.Lett. 105, 171601 (2010).

Contraction of Virasoro Algebra

The central Charge appears in the Virasoro Algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}n(n^2-1)\delta_{n+m}$$

So we are able to recover it in the process of contraction:

$$M_n = \lim_{c\to 0} c(L_n + \bar{L}_n)$$

$$l_n = \lim_{c\to 0}(L_n - \bar{L}_n)$$

Note in this section Capital C is the central charge and lower case c is the velocity of light.

Contraction of Virasoro Algebra

The new central charges after contraction are given by:

$$C_L = \lim_{c \to 0} \frac{1}{12}(c - \bar{c}) = 0$$

$$C_M = \lim_{c \to 0} \frac{c}{12}(c + \bar{c}) \to \frac{1}{4}$$

The centrally extended CGA algebra is then given:

$$[l, l] \sim l$$

$$[l, M] \sim M$$

$$[M, M] \sim 0$$

M and l are the so-called super-translations and super-rotations

Ali Hosseiny, Shahin Rouhani, Affine extension of Galilean conformal algebra in 2+ 1 dimensions JMP (2010)

Conformal Galilean Anomaly

For two-dimensional theories, one can show by demanding the Lorentz and scaling symmetry together with conservation of energy-momentum tensor and unitarity that :

$$\left\langle T^{\mu}{}_{\mu} \right\rangle = \frac{c}{24\,\pi}\,R$$

Where c is the central charge and R is the background curvature.

The question is: What is the form of non-relativistic conformal anomaly, for a system with Carroll symmetry ?

Can a flat manifold have a curved boundary?

Holographic Conformal Trace Anomaly

In order to achieve a holographic trace anomaly we need to choose a metric which does not tend to a flat boundary:
So choose the 3d asymptotically locally AdS Space-time in the BMS gauge (x_0 = constant hypersurface is a null surface);

$$ds^2 = A(u,r,\varphi)du^2 - 2e^{\beta(u,\varphi)}dudr + 2B(u,r,\varphi)dud\varphi + r^2 d\varphi^2$$

then take the limit $\mathrm{L} \to \infty$

Holographic Conformal Trace Anomaly

Which leads to the boundary metric:

$$ds^2{}_{boundary} = -\frac{G^2}{L^2}\, e^{4\beta(u,\varphi)} du^2 + G^2 d\varphi^2$$

This leads to the Ricci scalar (on two dimensional boundary)

$$R_2 = -\frac{4}{G^2}\, \Delta\beta$$

From which we can read the central charge.

Holographic Conformal Trace Anomaly

$$c = \frac{3L}{2\,G}$$

Thus making correspondence with our earlier results we have :

$$c_M = \frac{1}{4}$$

Note that β is dimensionless here, and corresponds to the operator φ introduced earlier $T = \partial^2\varphi$

Conformal Trace Anomaly

Finally :

$$\langle T + h \rangle \sim C_M \, R$$

In the above equation *T* is the stress component of the Energy-Momentum tensor and *h* is the energy density, as defined previously,

In the limit we essentially have a flat manifold and R is zero. Thus we <u>redrive</u> the traceless result that

$$T + h = 0$$

Bagchi, Arjun, and Reza Fareghbal. "BMS/GCA redux: towards flats pace holography from non-relativistic symmetries." Journal of High Energy Physics 2012.10 (2012): 1-30.
Fareghbal, Reza, and Ali Naseh. "Flat-space energy-momentum tensor from BMS/GCA correspondence. " Journal of High Energy Physics 2014.3 (2014): 1-12.

The importance of scale invariance lies in the c-theorem

There exists a function C(g), depending on the parameters of the theory {g} and scale, which is monotonically decreasing along the RG trajectory:

$$\frac{dC}{d\Lambda} = -\beta(g)^{\,2} \leq 0$$

and gives the beta functions :

$$\frac{\Lambda\, dg}{d\Lambda} = -\frac{\partial C}{\partial g} = \beta(g)$$

Critical Points correspond to Conformal Field Theories

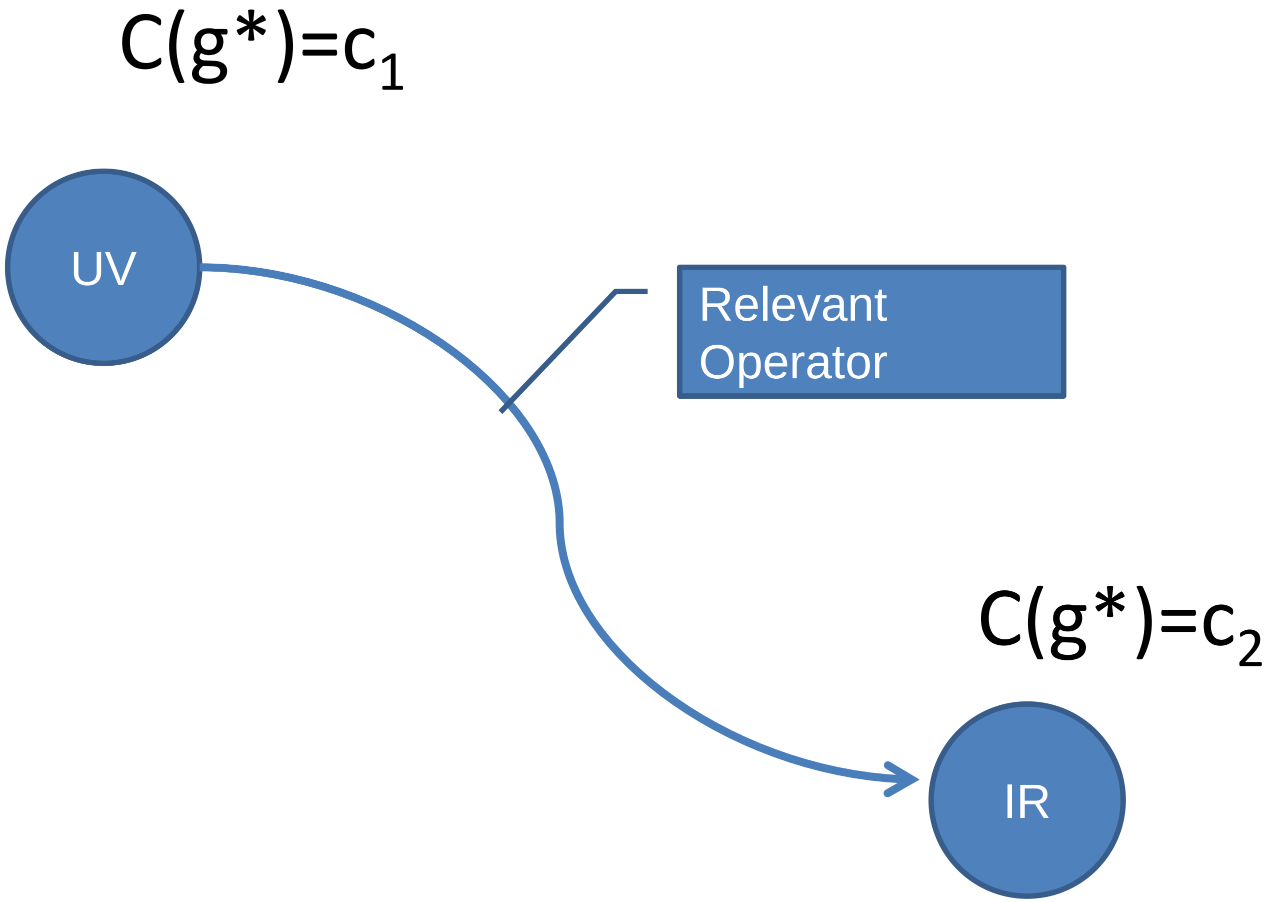
Thus the fixed points :g^*

$$\beta(g^*) = 0$$

Correspond to scale invariant theories hence CFTs.

Also other asymptotic behavior for the RG flow such as limit cycles and strange attractors are ruled out.

RG flow for the c-function



Central charge has a discrete spectrum

Lyapunov function

The straight forward way to look for a Lyapunov function is within the Holographic construct.

Consider the geometry :
$$ds^2 = -2dudr - \frac{1}{A^2L^2}(du^2 - d\varphi^2)$$

Then the function $C(r) = \frac{c}{A}$ Is monotonically increasing

Lyapunov function

$$\frac{d}{dr} C(r) \geq 0$$

note that for AdS solution $C(r) = c\, r$

Summary

- We showed that scale invariance leads to conformal invariance for Carrollian space-time

- Conformal Anomaly for the AdS/CFT correspondence can be calculated and is consistent with Carrollian space-time

- A Lyapunov function exists

Open Questions:

Why $c \to \infty$ contraction does not work ?

Repeat for Schrodinger symmetries .