# Wormholes: flare-out conditions

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# 1 Outline

- Morris-Thorne wormhole and flare-out conditions
- Spherically symmetric thin-shell wormholes
- Thin-shell wormhole supported by normal matter
- Example: The Hypocycloid
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## 2 Morris-Thorne wormhole and flare-out conditions

The general line element of the Morris-Thorne wormhole [Morris and Thorne, A. J. P. **56**,395(1988)]

$$ds^2 = -e^{-2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega^2 \tag{1}$$

in which $\Phi$ is red-shift function and $b$ is called shape-function. Einstein equations $(8\pi G = 1)$

$$G_\mu^\nu = T_\mu^\nu \tag{2}$$

implies

$$\rho = \frac{b'}{r^2}, \tag{3}$$

$$p_r = \frac{2\Phi'(r-b) - \frac{b}{r}}{r^2} \tag{4}$$

and

$$p_\theta = p_\phi = p_r + \frac{1}{2} r \left[ p_r' + (p_r + \rho)\Phi' \right]. \tag{5}$$

For a traversable wormhole, $b$ must satisfy certain conditions which are called Flare-out conditions. Let's look at the embedded spacetime at $t = const.$ and $\theta = \frac{\pi}{2}$ which is given by

$$ds_2^2 = \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\phi^2 = dr^2 + dz^2 + r^2 d\phi^2 \tag{6}$$

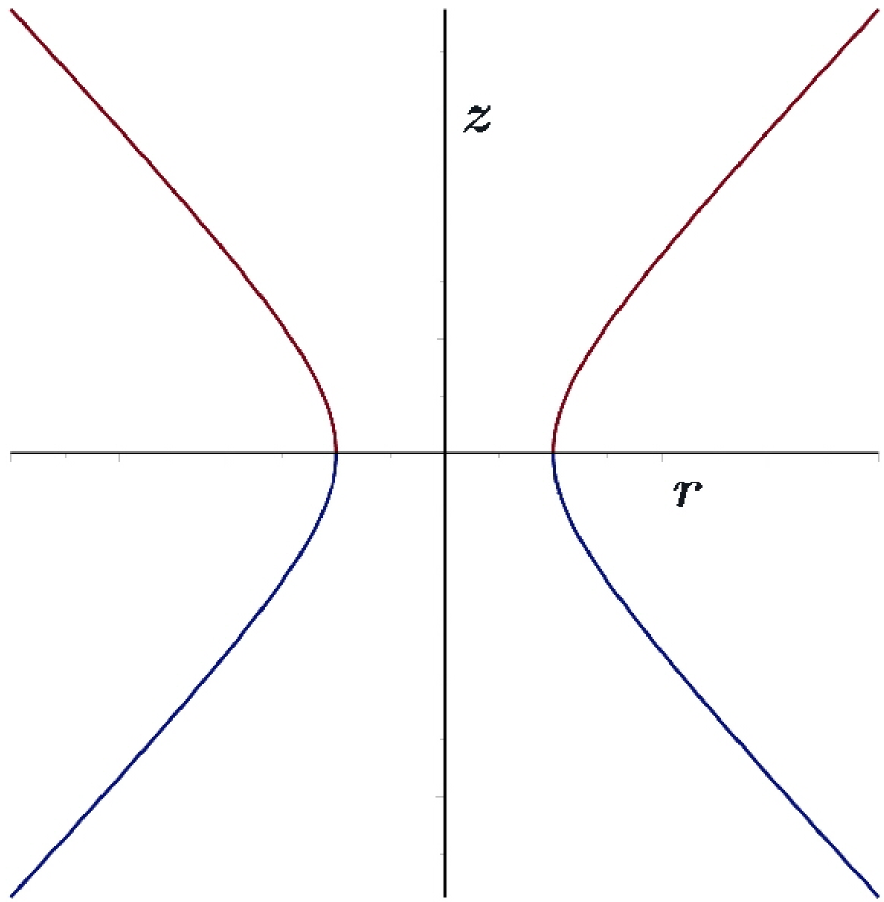
which implies

$$\frac{dz}{dr} = \frac{1}{\sqrt{\frac{r}{b} - 1}} \tag{7}$$

or

$$\frac{dr}{dz} = \sqrt{\frac{r}{b} - 1}. \tag{8}$$

At the throat, $\frac{dr}{dz} = 0$ and $\frac{d^2r}{dz^2} > 0$ which is seen from the Fig. 1. Hence, $\frac{dr}{dz} = 0$ at the throat implies $b\left(r = a\right) = a$ in which $r = a$ is the location of the throat while $\frac{d^2r}{dz^2} > 0$ implies $\frac{b - rb'}{2b^2\sqrt{\frac{r}{b} - 1}} > 0$ for $r > a$.


z
r

In summary the flare-out conditions are:

- $b \leq r$ for $a \leq r$
- $b\,(a) = a$
- $b' < \frac{b}{r}$

Now is time to check the WEC:

**i** $\rho \geq 0$

**ii** $\rho + p_r \geq 0$

**iii** $\rho + p_{\theta/\phi} \geq 0$

Simply one can see, for instance ii, can not be satisfied near the throat:

$$\rho + p_r = \frac{2\Phi' (r - b) + b' - \frac{b}{r}}{r^2} \tag{9}$$

which at the throat becomes

$$\rho + p_r = \frac{b' - \frac{b}{r}}{r^2} \tag{10}$$

which is clearly negative by considering the flare-out conditions.

In $2+1-$dimensions:

$$ds^2 = -e^{-2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\theta^2 \tag{11}$$

$$\rho = \frac{rb' - b}{2r^3}, \tag{12}$$

$$p_r = -\frac{\Phi'(b-r)}{r^2} \tag{13}$$

$$p_\theta = p_r + r\Phi'(\rho + p_r) \tag{14}$$

where the flare-out conditions prevents the energy to be positive i.e., $\rho < 0$.

# 3 Spherically Symmetric Thin-shell Wormholes

Let's consider a spherically symmetric manifold $\mathcal{M}$ [Visser, PRD **39**,3182(1989)]

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{15}$$

Next, we cut the part $r < a$ from $\mathcal{M}$ and the remaining part is called $\mathcal{M}^+$ which is an incomplete manifold. Now we make an identical copy of $\mathcal{M}^+$ and call it $\mathcal{M}^-$. Finally we paste $\mathcal{M}^+$ and $\mathcal{M}^-$ such that they are identified at the timelike hyperplane $\Sigma = r - a = 0$.

In order to glue the two incomplete manifold smoothly, one must apply the Israel junction conditions which are the Einstein equations on the shell whose induced line element is given by

$$ds^2_{\Sigma+} = ds^2_{\Sigma-} = ds^2_{\Sigma} = -d\tau^2 + a^2(\tau)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{16}$$

where $\tau$ is the proper time on the shell and

$$f(a)\dot{t}^2 - \frac{\dot{a}^2}{f(a)} = 1. \tag{17}$$

The extrinsic curvature of the shell in different side is given by

$$K^{\pm}_{ij} = n^{\pm}_{\gamma}\left(\frac{d^2x^{\gamma}}{dx^i dx^j} + \Gamma^{\gamma}_{\alpha\beta}\frac{dx^{\alpha}}{dx^i}\frac{dx^{\beta}}{dx^j}\right) \tag{18}$$

with the spacelike normal vector

$$n^{\pm}_{\gamma} = \left(\frac{\frac{d\Sigma}{dx^{\gamma}}}{\sqrt{g^{\alpha\beta}\frac{d\Sigma}{dx^{\alpha}}\frac{d\Sigma}{dx^{\beta}}}}\right)^{\pm}. \tag{19}$$

Israel conditions may be written as

$$\left[K_i^j\right] - [K]\,\delta_i^j = -S_i^j \tag{20}$$

in which $\left[K_i^j\right] = K_i^{j+} - K_i^{j-}$ and $[K] = \left[K_i^i\right]$. Irrespective of the matter source of the main manifold $\mathcal{M}$, $S_i^j = \left(-\sigma, P_\theta, P_\phi\right)$ is the energy momentum tensor of the matter on the shell such that in order to have the thin-shell wormhole physically acceptable $S_i^j$ must satisfy the energy conditions. The explicit expressions are given

$$\sigma = -\frac{4}{a}\sqrt{f\left(a\right) + \dot{a}^2}, \tag{21}$$

$$P_\theta = P_\phi = \frac{f'\left(a\right) + 2\ddot{a}}{\sqrt{f\left(a\right) + \dot{a}^2}} + \frac{2}{a}\sqrt{f\left(a\right) + \dot{a}^2}. \tag{22}$$

One observes that with $\sigma < 0$ the shell is supported by exotic matter.

In $2+1-$dimensional spacetime, the story is almost the same, except $\mathcal{M}$ is given by

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 \tag{23}$$

and

$$ds^2_{\Sigma+} = ds^2_{\Sigma-} = ds^2_{\Sigma} = -d\tau^2 + a^2(\tau)\,d\theta^2 \tag{24}$$

with

$$\sigma = -\frac{2}{a}\sqrt{f(a) + \dot{a}^2}, \tag{25}$$

$$P_\theta = P_\phi = \frac{f'(a) + 2\ddot{a}}{\sqrt{f(a) + \dot{a}^2}}. \tag{26}$$

# 4 Thin-shell wormhole supported by normal matter

We shall work in $2+1-$dimensions for the rest of this talk. [Mazharimousavi and Halilsoy, 2015]

In the standard construction of the thin-shell wormhole the hyperplane $r = a$ has represented a spherical shell (in $2+1-$dimension it is just a ring). We start with a flat spacetime

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 \tag{27}$$

and the throat to be deformed from a ring, such that $r = R(t, \theta)$. The induced metric becomes

$$ds_\Sigma^2 = -\left(1 - \dot{R}^2\right) dt^2 + \left(R^2 + R'^2\right) d\theta^2 + 2\dot{R}R' dt d\theta \tag{28}$$

in which a prime and a dot stand for derivative with respect to $\theta$ and $t$, respectively. One can show that

$$\dot{R}^2 = \frac{\left(\frac{\partial R}{\partial \tau}\right)^2}{\left(1 + \left(\frac{\partial R}{\partial \tau}\right)^2\right)}. \tag{29}$$

The energy momentum tensor on the shell has the following components

$$\sigma = \frac{2\left[\left(1-\dot{R}^2\right)\left(R'' - R - \frac{2R'^2}{R}\right) + \dot{R}R'\left(\dot{R}' - \frac{R'\dot{R}}{R}\right)\right]}{\left[\left(R'^2 + R^2\right)\left(1-\dot{R}^2\right) + \dot{R}^2 R'^2\right]\sqrt{1 + \left(\frac{R'}{R}\right)^2 - \dot{R}^2}} \tag{30}$$

$$P = \frac{-2\left[\left(1-\dot{R}^2\right)\left(\dot{R}' - \frac{R'\dot{R}}{R}\right) - \left(R'^2 + R^2\right)\ddot{R}\right]}{\left[\left(R'^2 + R^2\right)\left(1-\dot{R}^2\right) + \dot{R}^2 R'^2\right]\sqrt{1 + \left(\frac{R'}{R}\right)^2 - \dot{R}^2}} \tag{31}$$

$$S_t^{\theta} = \frac{2\left[\dot{R}R'\ddot{R} + \left(1-\dot{R}^2\right)\left(\dot{R}' - \frac{R'\dot{R}}{R}\right)\right]}{\left[\left(R'^2+R^2\right)\left(1-\dot{R}^2\right)+\dot{R}^2R'^2\right]\sqrt{1+\left(\frac{R'}{R}\right)^2-\dot{R}^2}} \tag{32}$$

and

$$S_{\theta}^{t} = \frac{2\left[\dot{R}R'\left(R''-R-\frac{2R'^2}{R}\right)-\left(R'^2+R^2\right)\left(\dot{R}'-\frac{R'\dot{R}}{R}\right)\right]}{\left[\left(R'^2+R^2\right)\left(1-\dot{R}^2\right)+\dot{R}^2R'^2\right]\sqrt{1+\left(\frac{R'}{R}\right)^2-\dot{R}^2}}. \tag{33}$$

Let's consider the throat to be static i.e., $R = R_0$, and $\dot{R} = \ddot{R} = 0$, therefore

$$\sigma_0 = \frac{2\left(R_0''-R_0-\frac{2R_0'^2}{R_0}\right)}{\left(R_0'^2+R_0^2\right)\sqrt{1+\left(\frac{R_0'}{R_0}\right)^2}}, \tag{34}$$

and

$$P_0 = S^t_\theta = S^\theta_t = 0. \tag{35}$$

Having only the energy density non-zero makes the energy conditions very simple given by

$$\sigma_0 \geq 0. \tag{36}$$

This is not surprising since the bulk spacetime is flat. Therefore in the static equilibrium, the only nonzero component of the energy-momentum tensor on the throat is the energy density $\sigma_0$. We note that the total matter supporting the wormhole is given by

$$U = \int_0^{2\pi} \int_0^{\infty} \sqrt{-g}\sigma\delta\left(r - R\right) dr d\theta = \int_0^{2\pi} R_0 \sigma_0 d\theta. \tag{37}$$

Now our task is to find $r = R = R_0\left(\theta\right)$ such that:

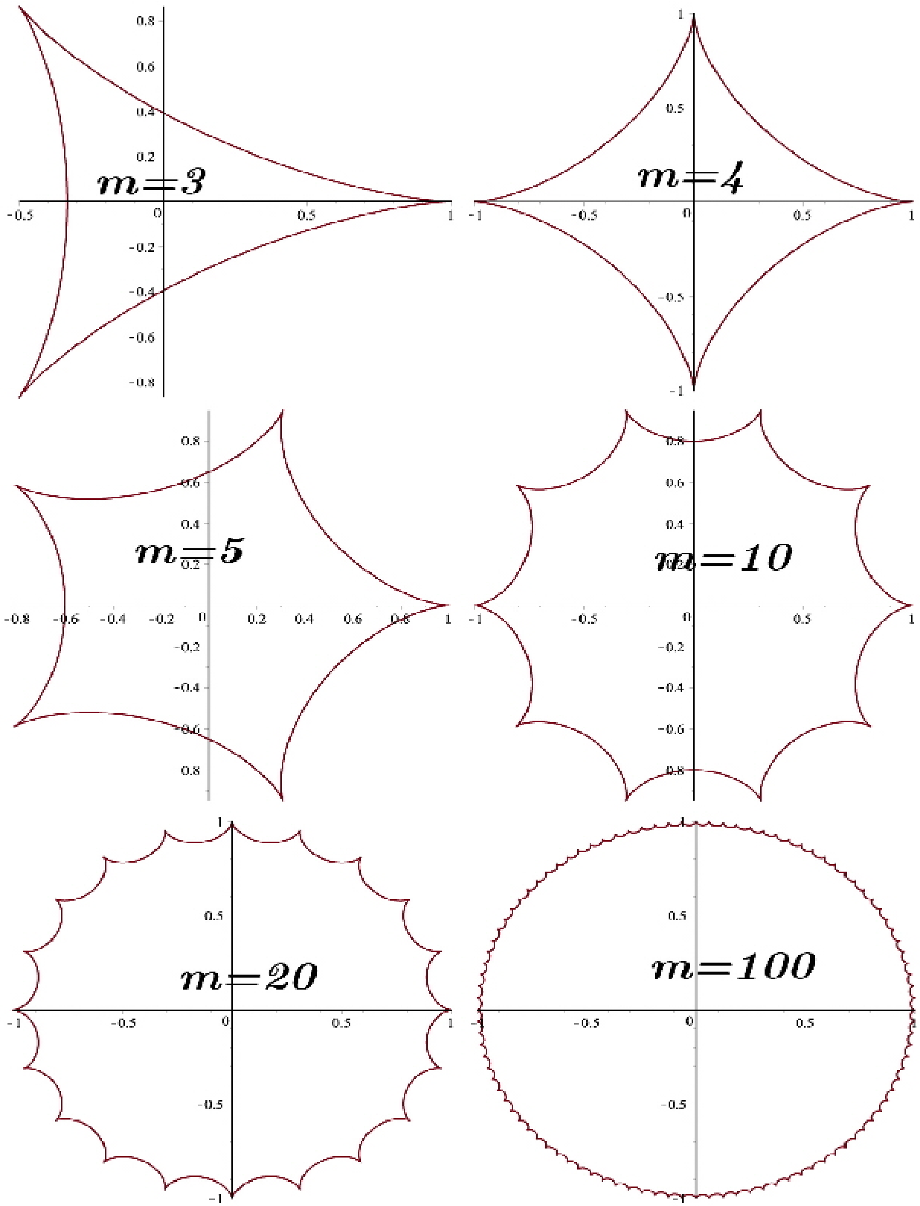
- $r = R = R_0(\theta)$ presents a closed timelike hyperplane
- $\sigma_0 \geq 0$ or $R_0'' - R_0 - \frac{2R_0'^2}{R_0} \geq 0$.

# 5 Example: The Hypocycloid

Definition: Hypocycloid is the curve generated by a rolling small circle inside a larger circle. This is a different version of the standard cycloid which is generated by a circle rolling on a straight line. The parametric equation of a hypocycloid is given by

$$\begin{aligned} x(\zeta) &= (B-b)\cos\zeta + b\cos\left(\frac{B-b}{b}\zeta\right) \\ y(\zeta) &= (B-b)\sin\zeta - b\sin\left(\frac{B-b}{b}\zeta\right) \end{aligned} \tag{38}$$

in which $x$ and $y$ are the Cartesian coordinates on the hypocycloid. $B$ is the radius of the larger circle centered at the origin, $b\,(< B)$ is the radius of the smaller circle and $\zeta \in [0, 2\pi]$ is a real parameter. Here if one considers $B = mb$, where $m \geq 3$ is a natural number, then the curve is closed and it possesses $m$ singularities / spikes. In Fig. 2 we plot (38) for different values of $m$ with $B = 1$.


m=3
m=4
m=5
m=10
m=20
m=100

Without loss of generality we set $B = 1$ and $b = \frac{1}{m}$ and express $\sigma$ as a function of $\zeta$. For this we parametrize the equation of the throat as

$$\begin{aligned} R &= R(\zeta) = \sqrt{x(\zeta)^2 + y(\zeta)^2} \\ \theta &= \theta(\zeta) = \tan^{-1}\left(\frac{y(\zeta)}{x(\zeta)}\right). \end{aligned} \tag{39}$$

Using the chain rule one finds

$$R' = \frac{dR}{d\theta} = \frac{\dot{R}}{\dot{\theta}} \tag{40}$$

and

$$R'' = \frac{d^2R}{d\theta^2} = \frac{\ddot{R}\dot{\theta} - \dot{R}\ddot{\theta}}{\dot{\theta}^3} \tag{41}$$

which implies

$$\sigma = \frac{1}{4\pi}\frac{R\ddot{R}\dot{\theta} - R\dot{R}\ddot{\theta} - R^2\dot{\theta}^3 - 2\dot{\theta}\dot{R}^2}{\left(\dot{R}^2 + R^2\dot{\theta}^2\right)^{\frac{3}{2}}} \tag{42}$$

where a dot stands for the derivative with respect to the parameter $\zeta$. Consequently the total matter is given by

$$U = \int_0^{2\pi} u d\zeta \tag{43}$$

where $u = R\sigma\dot{\theta}$ is the energy density per unit parameter $\zeta$. Note that for the sake of simplicity we dropped the sub-index 0 from the quantities calculated at the throat. Particular examples of calculations for the energy $U$ are given as follows.

### 5.1 $m = 3$

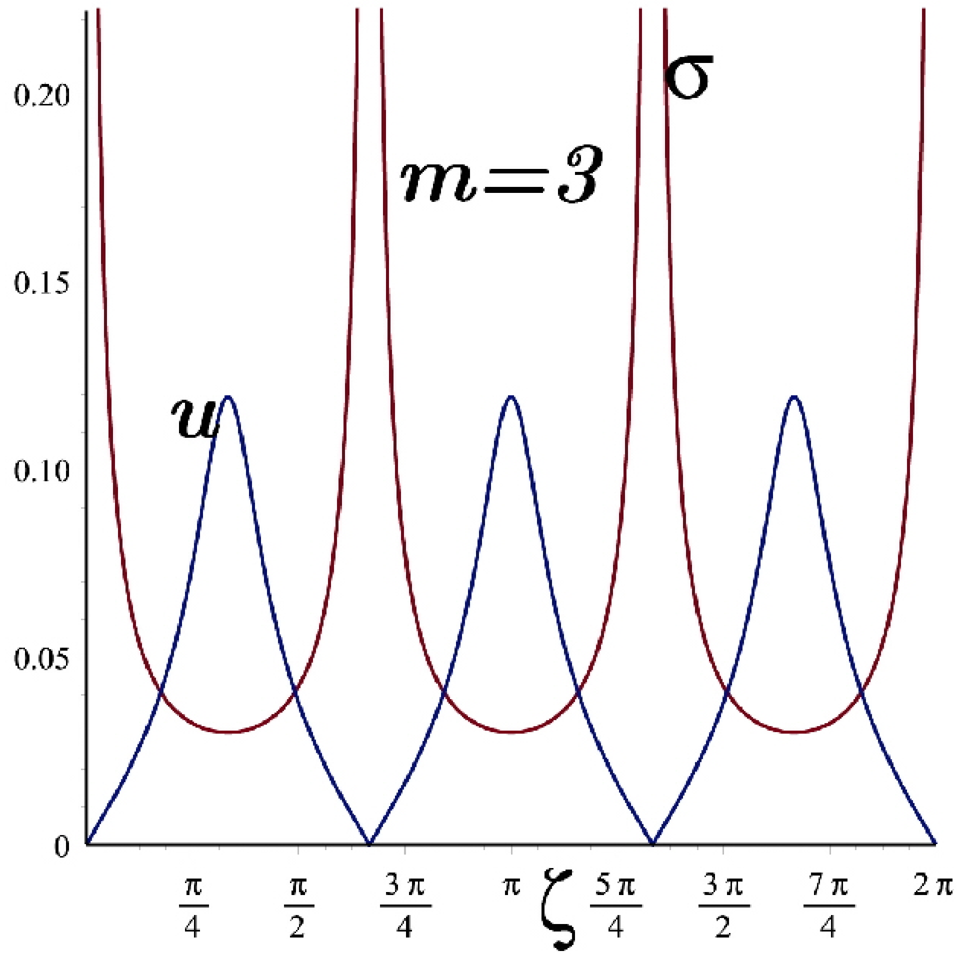
The first case which we would like to study is the minimum index for $m$ which is $m = 3$. We find that

$$\sigma = \frac{3\sqrt{2}}{32\pi\sqrt{(1+2\cos\zeta)^2(1-\cos\zeta)}} \tag{44}$$

which is clearly positive everywhere. Knowing that the period of the curve (38) is $2\pi$ we find that $\sigma$ is singular at the possible roots of the denominator i.e., $\zeta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$. We note that although $\sigma$ diverges at these points the function that must be finite everywhere is $u$ which is given by

$$u = \frac{3\sqrt{2}\sqrt{(1+2\cos\zeta)^2(1-\cos\zeta)}}{16\pi\sqrt{5-12\cos\zeta+16\cos^3\zeta}}. \tag{45}$$

The situation is in analogy with the charge density of a charged conical conductor whose charge density at the vertex of the cone diverges while the total charge remains finite. In Fig. 3 we plot $\sigma$ and $u$ as a function of $\zeta$ which clearly implies that $u$ is finite everywhere leading to the total finite energy $U_3 = 0.099189$.



0.20
0.15
0.10
0.05
0
σ
m=3
u
π/4
π/2
3π/4
π
ζ
5π/4
3π/2
7π/4
2π

### 5.2 $m = 4$

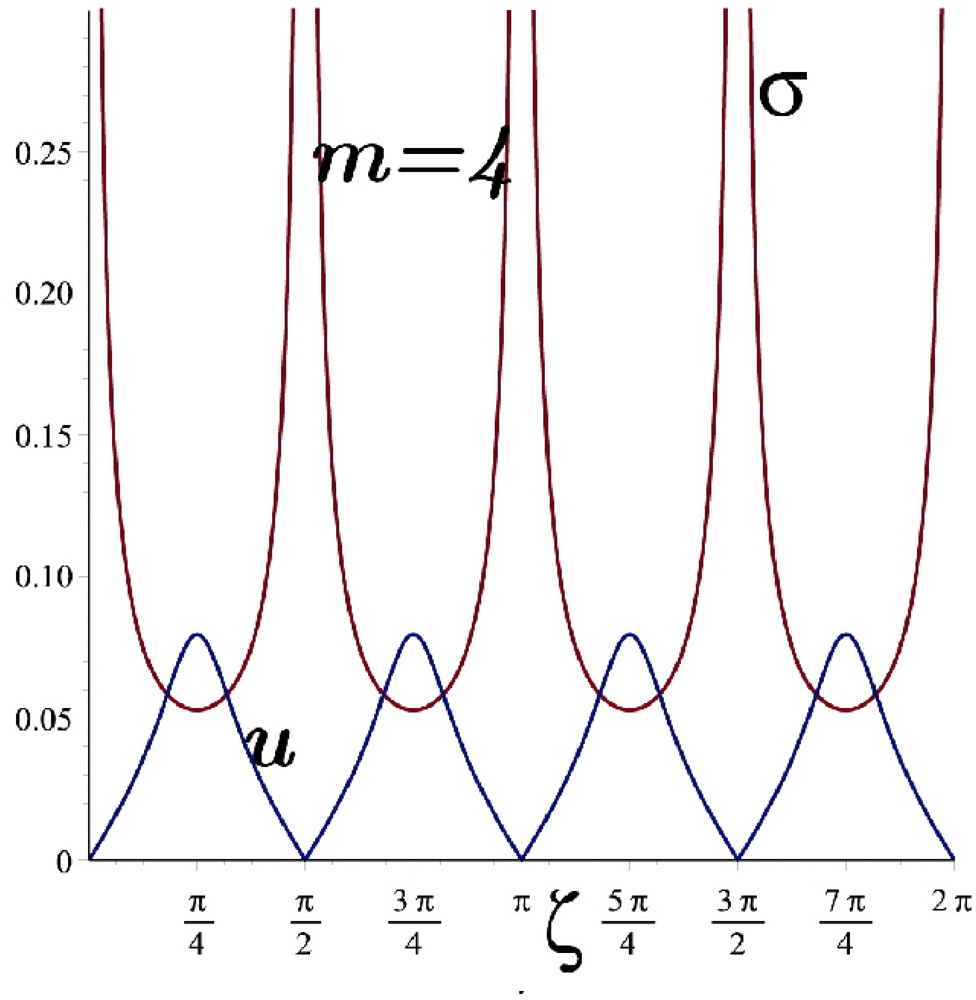
Next, we set $m = 4$ where one finds

$$\sigma = \frac{1}{6\pi\sqrt{\sin^2(2\zeta)}}, \tag{46}$$

and

$$u = \frac{\sqrt{\sin^2(2\zeta)}}{8\pi\sqrt{1 - 3\cos^2\zeta + 3\cos^4\zeta}}. \tag{47}$$

Fig. 4 depicts $\sigma$ and $u$ in terms of $\zeta$ and similar to $m = 3$, we find $\sigma > 0$ and $u$ finite with the total energy given by $U_4 = 0.24203$.



0.25
0.20
0.15
0.10
0.05
0
m=4
σ
u
π/4
π/2
3π/4
π
ζ
5π/4
3π/2
7π/4
2π

As one observes $U_4 > U_3$ which implies that adding more cusps to the throat increases the energy needed. This is partly due to the fact that the total length of the hypocycloid is increasing as $m$ increases such that $\ell_m = \frac{8(m-1)}{m}$ with $B = 1$. This pattern goes on with $m$ larger and in general

$$u = \frac{(m-2)^2 \sqrt{(\cos\zeta - \cos(m-1)\zeta)^2}}{8\pi\sqrt{2}\sqrt{\Psi}} \tag{48}$$

where

$$\begin{aligned}\Psi = m^2 - 2(m-1)\cos^2(m-1)\zeta \\ -(m-2)^2\cos\zeta\cos(m-1)\zeta- \\ m^2\sin(m-1)\zeta\sin\zeta - 2(m-1)\cos^2\zeta.\end{aligned} \tag{49}$$

Table 1 shows the total energy $U_m$ for various $m$. We observe that $U_m$ is not bounded from above (with respect to $m$) which means that for large $m$ it diverges as $U_m \approx \frac{m}{2\pi}$.

Therefore to stay in classically finite energy region one must consider $m$ to be finite.

| $m$ | 3 | 4 | 5 | 10 | 50 | 100 |
|---|---|---|---|---|---|---|
| $U_m$ | 0.099189 | 0.24203 | 0.39341 | 1.1767 | 7.5351 | 15.492 |

# 6 Conclusion

$>$ The flare-out conditions prevents wormholes to be physical.

$>$ Thin-shell wormholes allows to locate matter source at the location of the throat.

$>$ Changing the geometry of the throat in thin-shell wormholes may let the energy condition to be satisfied.

$>$ In $2+1-$dimensions, a very simple example has shown that such physical wormholes are possible.

$>$ In $3+1-$dimensions and higher a similar formalism works.

# References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[3] S. Habib Mazharimousavi and M. Halilsoy, Eur. Phys. J. C **74**, 3067 (2014).