Scattering amplitudes computation in Quantum Chromodynamics

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The Dawn of a New Era

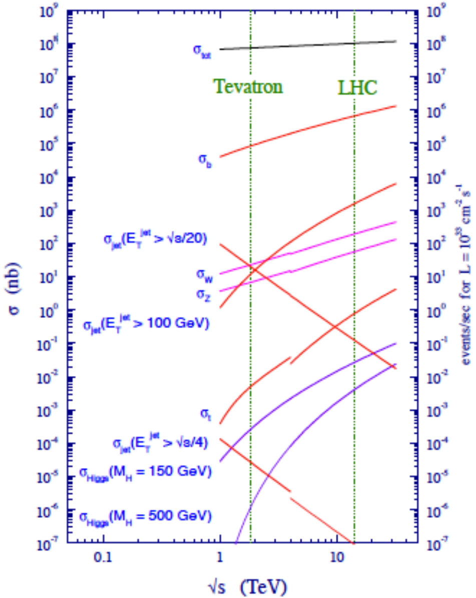
- The July 2012 announcement of the discovery of a Higgs-like boson at CERN by ATLAS and CMS completed our discovery of the Standard $SU(3) \times SU(2) \times U(1)$ Model.
- Captures three of the four known forces.
- Misses dark matter, most of the matter in the universe.
- Is the Standard Model just an effective energy theory?

Experiments

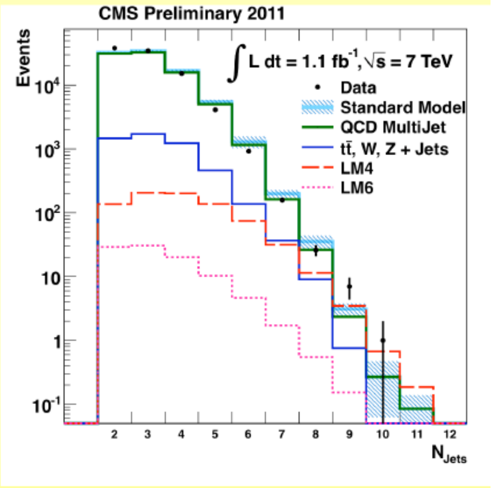
- Need input from experiments.
- Direct searches from physics beyond the Standard Model.
- Indirect searches
  - Precision measuments of the Higgs, of the top quark, of electroweak vector bosons;
  - Rare decays: K, D, B;
  - Muon magnetic moment;
  - Neutrino mixing.
- Theory complements: precision calculations of signal and backgrounds.

LHC DATA dominated by jets

Proton (anti)-Proton Standard Model cross sections

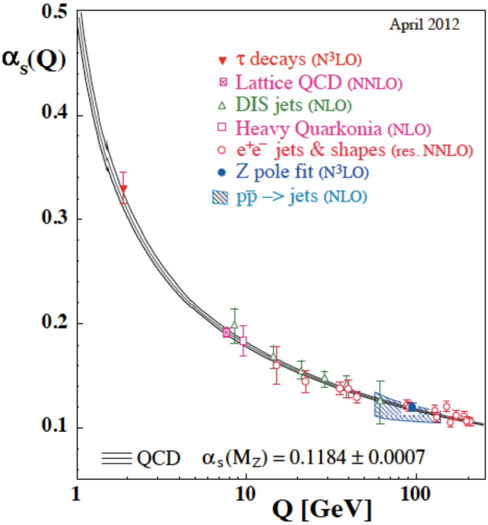


Jets are ubiquitous



Even by the techniques of amplitudes calculation we are going to illustrate to make prediction about events with 10 jets is extremely challenging.

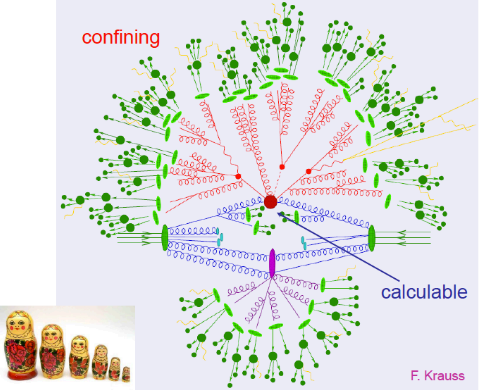
Complexity is due to QCD

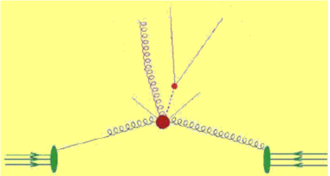


(Betkhe)

Strong coupling is not small: $\alpha_S(M_Z) \approx 0.12$ and running is important

- events have multiplicity of hard clusters (jets);
- each jet has a high multiplicity of hadrons;
- higher-order perturbative corrections are important.


confining
calculable
F. Krauss



Fixed Order Calculation

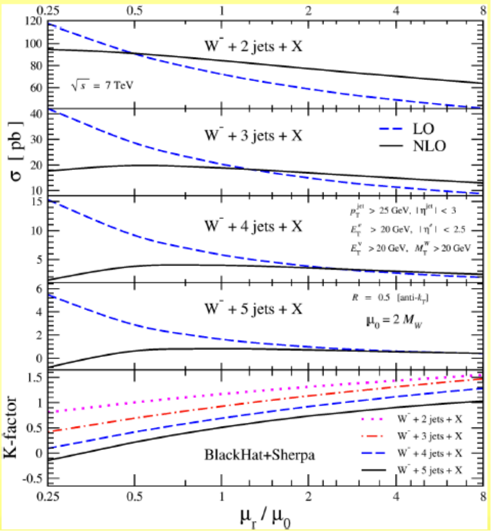
- Simplify to essentials:
  - Focus on jets.
  - Numerical jet programs:general observables.
  - Systematic to higher order/high multiplicity in perturbation theory.
  - Parton level, approximate jet algorithm; match detector events only statistically.
- Every infrared sensible observable has a perturbative expansion in α_S

$$\frac{d\sigma^{\mathrm{W+3jet}}}{dp_T^{\mathrm{2ndjet}}} = \alpha_S^3(\mu)\frac{d\sigma^{\mathrm{LO}}}{dp_T^{\mathrm{2ndjet}}} + \alpha_S^4(\mu)\frac{d\sigma^{\mathrm{NLO}}}{dp_T^{\mathrm{2ndjet}}}(\mu) + \alpha_S^5(\mu)\frac{d\sigma^{\mathrm{NNLO}}}{dp_T^{\mathrm{2ndjet}}}(\mu)$$

LO=Leading Order

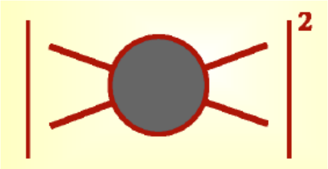
Theory of many Jets

- Want quantitative predictions.
- Renormalization scale needed to define α_S; factorization scale to separate long distance physics.
- Physical observables should be independent on scales; truncated perturbation theory is not.
- LO has large dependence.
- NLO reduces this dependence.
- NLO importance grows with increasing number of jets.
- Expect predictions reliable to $10 - 15\%$.
- $< 5\%$ predictions will require NNLO.

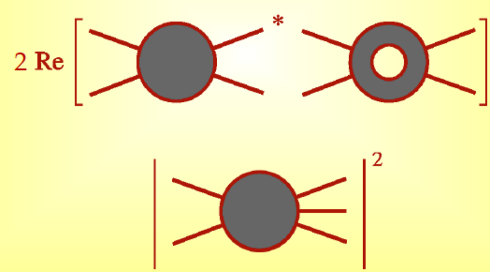

W- + 2 jets + X
$\sqrt{s}$ = 7 TeV
W- + 3 jets + X
LO
NLO
σ [pb]
W- + 4 jets + X
p_T^{jet} > 25 GeV, $|\eta^{jet}|$ < 3
E_T^e > 20 GeV, $|\eta^e|$ < 2.5
E_T^ν > 20 GeV, M_T^W > 20 GeV
W- + 5 jets + X
R = 0.5 [anti-k_T]
$\mu_0 = 2\,M_W$
K-factor
BlackHat+Sherpa
W- + 2 jets + X
W- + 3 jets + X
W- + 4 jets + X
W- + 5 jets + X
μ_r / μ_0

Ingredients for NLO Calculations

- Short distance matrix elements to 2-jet production at leading order:tree level amplitudes.

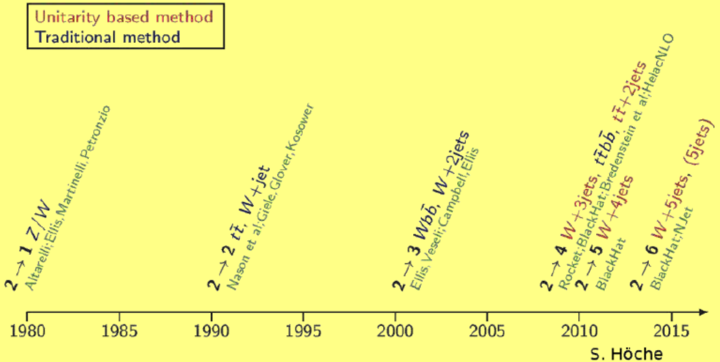


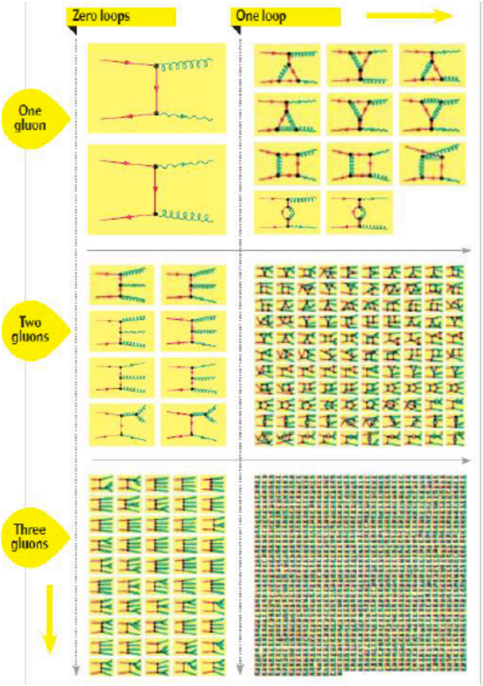
- Short distance matrix elements to 2-jet production at next-to-leading order:tree level + one loop amplitudes+ real emission



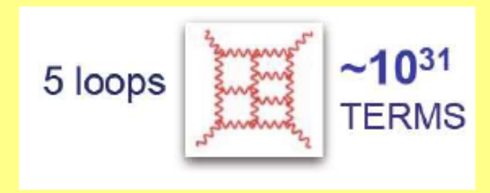
- Singular behaviour of tree level amplitudes, integrals, initial state collinear behaviour.
- NLO parton distribution
- General framework for numerical virtual cancellations (Catani-Seymour subtraction is most popular) and its automatation.

The NLO revolution




Zero loops
One loop
One gluon
Two gluons
Three gluons

- Huge number of diagrams in calculations of interest - factorial growth with number of legs and loops;
- $2 \to 6$ jets:34300 tree diagrams with $\sim 2.5 \cdot 10^7$ terms, $\sim 2.9 \cdot 10^6$ one-loop diagrams with $\sim 1.9 \cdot 10^{10}$ terms;
- In gravity it is even worse



Results are simple!

- Parke-Taylor formula for the tree level amplitude A^{MHV}

$$\imath \frac{< m_1 m_2 >^4 \ \delta^4(\sum_i K_i)}{< 12 >< 23 >< 34 > \ ...... < (n-1)n >< n1 >}$$

Parke, Taylor; Mangano, Parke; Xu
The simplicity is enhanced by the use of Lorentz invariant spinor products

$$< 12 > \sim \sqrt{2K_1 \cdot K_2}$$

Even simpler in N=4 Supersymmetric theory

- Nair-Parke-Taylor formula for MHV-class amplitudes

$$\imath \frac{\delta^{4|8}(\sum_i \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}} | \sum_i \lambda_i^{\alpha} \eta_i^{A})}{< 12 >< 23 >< 34 > ...... < (n-1)n >< n1 >}$$

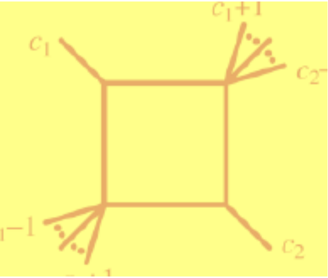
where the two component helicity spinors are roughly

$$\lambda_i^{\alpha} \sim \sqrt{K_i^{\mu}}$$

Answers are simple at Loop level Too

One loop in Susy N=4

$$A^{\rm tree}(1^+, ..., i^- ....j^-, ...n^+) \times \sum_{\rm easy\,two\,masses} {\rm Box}\frac{1}{2}({\rm its\ denominator})$$



All-n QCD amplitudes for MHV configurations on a few pages of Phys. Rev. D.

Calculation is a Mess

- Diagrams inside involve unphysical states
- Each diagram does not respect the symmetry of the theory ("not gauge invariant")- huge cancellations of gauge non invariant redundant parts are to blame (exacerbated by algebra).
- There is almost no information in any given diagram.
- A new subfield **AMPLITUDES** has been created where the convergence of gauge theories, string theories and integrability provides the needed tools to calculate large class of amplitudes in gauge theories, (in principle) with infinite number of legs as well as a wealth of data for further studies.

On shell methods

- All physical quantities computed
  - From basic interaction amplitudes: A_3^{tree}
  - Using only information from physical on-shell states
  - Avoid size explosion of intermediate terms due to unphysical states
  - Without need for a Lagrangian
- Properties of amplitudes becomes tools for calculating
  - Kinematics
    - Spinor variables
  - Underlying field theory
    - Integral basis
  - Factorization
    - On-shell recursion relation (BCFW) for tree level amplitudes
    - Control infrared divergencies in real-emission contribution to higher order calculation
  - Unitarity
    - Unitarity and generalized unitarity for loop calculation.

Integral Basis

- At one loop
  - Tensor reductions *à* la Brown, Feynman and Passarino, Veltman.
  - Gram determinant identities
  - Boxes, triangles, bubbles and tadpoles: Feynman integrals are expressible in terms of logarithms, dilogarithms, rational functions of invariants.
- At higher loops
  - Tensor reduction and Gram determinant identities
  - Irreducible numerators: integration by parts *à* la Chetyrkin-Tkachov
  - Laporta algorithm
  - AIR (Anastasiou-Lazopoulos), FIRE (Smirnov, Smirnov), Reduze (Manteuffel, Studerus), LiteRed (Lee).
  - Four dimensional basis: integrals up to 4L propagators.

BCFW On-Shell recurvive relations
Britto, Cachazo, Feng, Witten (2005)

- Define a shift $[j, l >$ of spinors by a complex parameter z

$$|j^- > \to |j^- > -z|l^- > \quad |l^+ > \to |l^+ > +z|j^+ >$$

which defines a z continuation of the amplitude $A(z)$

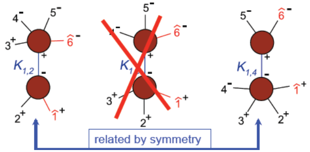
- Assume $A(z) \to 0$ as $z \to \infty$

$$\oint_{C_\infty} \frac{A(z)}{z} dz = 0 \Rightarrow A(z=0) = -\sum_\alpha Res_{z \to z_\alpha} \frac{A(z)}{z}$$

Poles in z come from kinematics poles in amplitudes

$$A(z) = \sum_\alpha \frac{c_\alpha}{z - z_\alpha}$$

The sum over residues gives the on-shell recursive relations.
For a six gluon example instead of the 220 diagrams there are just 3 BCFW diagrams.



bringing to the simple final form

$$\boxed{\begin{aligned}-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) \;=\; & \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} \\ & + \frac{\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}\end{aligned}}$$

NLO Revolution: On shell methods

Master equation
A dimensionally regularized amplitude in $D = 4 - 2\epsilon$ can be expanded as

$$\mathrm{Ampl} = \sum_{j \in \mathrm{basis}} c_j \mathrm{Int_j} + \mathrm{Rational}$$

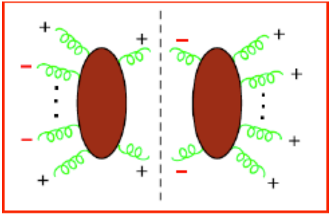
$\mathrm{Int_j}$ are the elements of the known integral basis
c_j are rational functions of the spinors of the external legs. Those coefficients can be obtained by generalized unitarity in $D = 4$.
The rational remainder (together with the coefficients c_j) can be obtained by generalized unitarity in $D = 4 - 2\epsilon$ dimensions (A.R.F., P.Mastrolia, E.Mirabella, W.J. Torres).

Unitarity

- Conservation of probability.
- At the diagram or amplitude level corresponds to Cutkosky rule: "cut" a pair of propagators.

$$\frac{1}{\ell^2 - m^2 + \imath 0} \to -2\pi\imath\delta_+(\ell^2 - m^2) \equiv -2\pi\imath\delta(\ell^2 - m^2)\theta(\ell^0).$$

- Reconstruct coefficients from the cuts-which are tree amplitudes.

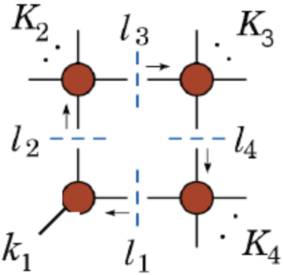


- No loop diagrams involved.

Generalized unitarity

- "Cut" more propagators with appropriate contour integration.
- Each contour integration imposes an "on-shell" condition.
- For the box integral four on-shell conditions freeze the loop momentum completely. For a massless box configuration

$$\ell^2 = 0 \quad -2\ell\cdot k_1 + k_1^2 = 0 \quad -2\ell\cdot K_2 + K_{12}^2 - k_1^2 = 0 \quad 2\ell\cdot K_4 + K_4^2 = 0$$



- Solutions are complex momenta!

- Coefficients expressed in terms of tree amplitudes evaluated at these momenta.

$$\text{Box coefficient} = \frac{1}{2} \sum_{\text{solutions}} \sum_{\text{species}} \sum_{\text{helicities}} \prod_{J} A_J^{\text{tree}}$$

- No algebraic reductions needed: suitable for pure numerics.
- One loop Susy N=4 amplitudes contain only boxes, due to Susy cancellations of loop momenta in the numerator.

The Easy Two-Mass box

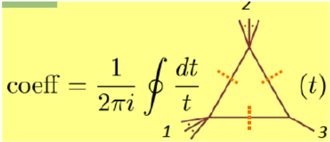
Let $k_1^2 \neq 0$ and $k_3^2 \neq 0$ and $s = (k_1 + k_3)^2$ and $t = (k_1 + k_4)^2$

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2(\ell - k_1)^2(\ell - k_1 - k_2)^2(\ell + k_4)^2} =$$
$$\frac{2c_\Gamma}{st - k_1^2 k_3^2} \frac{1}{\epsilon^2} \left[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-k_1^2)^{-\epsilon} - (-k_3^2)^{-\epsilon}\right]$$
$$-\frac{2c_\Gamma}{st - k_1^2 k_3^2} \left[\mathrm{Li}_2\left(1 - \frac{k_1^2}{s}\right) + \mathrm{Li}_2\left(1 - \frac{k_1^2}{t}\right)\right.$$
$$+\mathrm{Li}_2\left(1 - \frac{k_3^2}{s}\right) + \mathrm{Li}_2\left(1 - \frac{k_3^2}{t}\right)$$
$$\left.-\mathrm{Li}_2\left(1 - \frac{k_1^2 k_3^2}{st}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right)\right] + O(\epsilon)$$

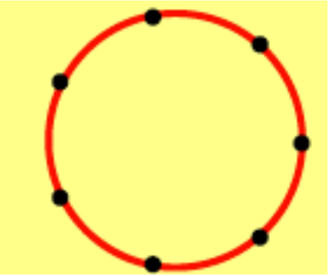
Dilogarithm $\mathrm{Li}_2(x) = -\int\limits_0^x \frac{\log(1-t)}{t} = \sum\limits_{k=0}^{\infty} \frac{z^k}{k^2}$

Triangle cuts

Unitarity leaves one degree of freedom in triangle integrals.
Coefficients are residues at ∞

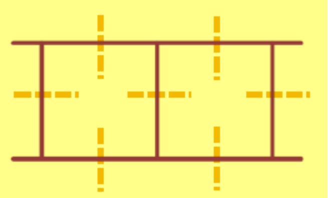


Evaluate numerically using a Fourier discrete projection (exact!)

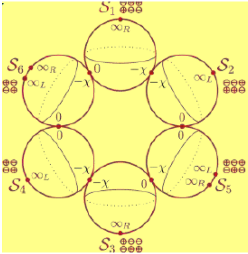


Higher loops

- Same master equation
- Formulas for coefficients still under development



- Connections with algebraic geometry (with Mastrolia, Mirabella, Torres)



Conclusions

- NLO calculations are the first step to precision theory at LHC.
- On shell methods have allowed to push these calculations to high multiplicities.
- Strong foundation for increasing precision and reach to match upcoming experimental improvements.