How far can we stretch our theory?

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Content:

▪ Short reminder of the spontaneous ElectroWeak symmetry breaking
▪ One missing piece: the mass of the Brout-Englert-Higgs scalar
▪ Theoretical constraints on the Higgs boson mass
  □ Pertubativity or unitarity constraint
  □ Triviality bound and stability bound
  □ Fine-tuning
▪ Methods can be applied to models beyond the Standard Model
▪ What about more then one Higgs doublet…

http://w3.iihe.ac.be/~jdhondt/Website/BeyondTheStandardModel.html
Spontaneously broken QED theory

Let us illustrate the “Higgs” mechanism with a massive U(1) theory before going to the symmetry group SU(2)\textsubscript{L} x U(1)\textsubscript{Y}. The Lagrangian of QED is:

\[ \mathcal{L}_{QED} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \overline{\psi} (i \gamma^\mu D_\mu - m) \psi \]

This is invariant under the U(1) gauge transformation

\[ \psi \rightarrow e^{-i \alpha(x)} \psi \]

\[ A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \alpha(x) \]

Now we wish to give the photon a mass by adding the term

\[ \mathcal{L}_{mass} = \frac{m_A^2}{2} A_\mu A^\mu \]

Which breaks the initial U(1) gauge symmetry. Hence need to invoke a mechanism which introduces a mass without breaking the symmetry.
Spontaneously broken QED theory

Introduce a complex scalar field $\Phi$ as

$$\mathcal{L} = \mathcal{L}_{QED} + (D_\mu \Phi)^* (D^\mu \Phi) - V(\Phi)$$

with the potential $V$ defined as

$$V(\Phi) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4$$

which is symmetric under the transformation $\Phi \rightarrow -\Phi$.

We can choose a parametrization as

$$\Phi = \frac{1}{\sqrt{2}} \phi(x) e^{i\xi(x)}$$

where both fields $\phi$ and $\xi$ are real fields.

The potential becomes

$$V(\Phi) = \frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4$$

where the Higgs self-coupling term should be positive ($\lambda > 0$) to get a potential bound from below. When $\mu^2 < 0$ a non-zero vacuum expectation value is obtained.

$$<0|\phi^2|0> = \phi_0^2 = \frac{\mu^2}{\lambda} = v^2$$
Spontaneously broken QED theory

Therefore we can normalize the field $\xi(x)$ as $\frac{\xi(x)}{\phi_0}$.

We can choose the unitary gauge transformation

$$\alpha(x) = -\frac{\xi(x)}{\phi_0}$$

and then $\Phi$ becomes real-valued everywhere. The kinetic term in the Lagrangian becomes

$$(D_\mu \Phi)^*(D^\mu \Phi) \rightarrow \frac{1}{2} \partial_\mu \psi \partial^\mu \psi + \frac{e^2}{2} A_\mu A^\mu \psi^2$$

The Lagrangian can be expanded around its minimum $\phi_0$ by introducing a degree of freedom $h$ (a new field). The potential becomes

$$V(\phi \rightarrow \phi_0 + h) = -\frac{m_h^2}{2} h^2 - \frac{\mu'}{3!} h^3 - \frac{\eta}{4!} h^4$$

with $m_h^2 = 2\lambda \phi_0^2$ and $\mu' = \frac{3m_h^2}{\phi_0}$ and $\eta = 6\lambda = 3\frac{m_h^2}{\phi_0^2}$.
Spontaneously broken QED theory

The kinetic term becomes
\[
\frac{1}{2} \partial_{\mu} (\phi_0 + h) \partial^{\mu} (\phi_0 + h) + \frac{e^2}{2} A_\mu A^\mu (\phi_0 + h)^2
\]
and with \( \partial_\mu \phi_0 = 0 \) this becomes
\[
e^2 \frac{A_\mu A^\mu \phi_0^2}{2} + e^2 A_\mu A^\mu \phi_0 h + \frac{e^2}{2} A_\mu A^\mu h^2 + \frac{1}{2} (\partial_{\mu} h) (\partial^{\mu} h)
\]
where the first term provides a mass to the photon \( m_A^2 = e^2 \phi_0^2 \),
the second term gives the interaction strength of the coupling A-A-h,
the third term the interaction strength of the coupling A-A-h-h

In the new potential term \( V(\phi_0 + h) \) also cubic terms appear which break the reflexion symmetry \( \phi \rightarrow -\phi \).

This U(1) example is the most trivial example of a spontaneous broken symmetry.
Spontaneously broken SU(2)xU(1) theory

The bosonic part of the Lagrangian is

\[ \mathcal{L}_{\text{bosonic}} = |D_\mu \Phi|^2 - \mu^2 |\Phi|^2 - \lambda |\Phi|^4 - \frac{1}{4} B_{\mu \nu} B^{\mu \nu} - \frac{1}{4} W^a_{\mu \nu} W^{a \mu \nu} \]

with $\Phi$ a doublet field consisting out of two complex scalar fields or components

\[ \Phi = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} \]

We need at least 3 massive gauge bosons, hence need at least 2 complex fields (cfr. Goldstone theorem).

\[ D_\mu \Phi = \left( \partial_\mu + ig \frac{\tau^a}{2} W^a_\mu + ig' \frac{Y}{2} B_\mu \right) \Phi \]
\[ B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \]
\[ W^a_{\mu \nu} = \partial_\nu W^a_\mu - \partial_\mu W^a_\nu - gf^{abc} W^b_\mu W^c_\nu \]

with $\tau^a$ the Pauli matrices and $f^{abc}$ the structure constants of the SU(2)$_L$ group.
Spontaneously broken SU(2)xU(1) theory

The $B_\mu$ field corresponds to the generator $Y$ of the $U(1)_Y$ group and the three $W_\mu^a$ fields to the generators $T^a$ of the $SU(2)_L$ group.

$$T^a = \frac{1}{2} T^a$$

$$\left[ T^a, T^b \right] = i f^{abc} T^c$$

$$Tr \left[ T^a T^b \right] = \frac{\delta_{ab}}{2}$$

When $\mu^2 < 0$ the vacuum expectation value of $\overline{\Phi}$ is non-zero.

$$\langle 0 | \overline{\Phi} | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{with} \quad v = \sqrt{-\frac{\mu^2}{\lambda}}$$

The VEV will carry the hypercharge and the weak charge into the vacuum, but the electric charge remains unbroken, hence $Q = T^3 + \frac{Y}{2}$ and we break $SU(2)_L \times U(1)_Y$ to $U(1)_Q$ with only one generator.

Expending the terms in the Lagrangian around the minimum of the potential gives

$$\overline{\Phi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$
Spontaneously broken SU(2)\times U(1) theory

We obtain

\[ |D_\mu \Phi|^2 \to \frac{1}{2} (\partial_\mu h)^2 + \frac{1}{8} g^2 (v + h)^2 |W^{(1)}_\mu + iW^{(2)}_\mu|^2 + \frac{1}{8} (v + h)^2 |gW^{(3)}_\mu - g'B_\mu|^2 \]

and define the following fields

\[ W^{\pm}_\mu = \frac{1}{\sqrt{2}} \left( W^{(1)}_\mu \mp iW^{(2)}_\mu \right) \]
\[ Z_\mu = \frac{gW^{(3)}_\mu - g'B_\mu}{\sqrt{g^2 + g'^2}} \]
\[ A_\mu = \frac{gW^{(3)}_\mu + g'B_\mu}{\sqrt{g^2 + g'^2}} \]

which we can transform into expressions for \( B_\mu \) and \( W^{(i)}_\mu \) and put this in the above equation for \( |D_\mu \Phi|^2 \) and isolate the Higgs boson interaction terms

\[ \mathcal{L}_{Higgs\ int} = \left( m^2_W W^{\pm}_\mu W^{\mp}_\mu + \frac{m^2_Z}{2} Z_\mu Z^\mu \right) \left( 1 + \frac{h}{v} \right)^2 - \frac{m^2_h}{2} - \frac{\xi}{3!} h^3 - \frac{\eta}{4!} h^4 \]
Spontaneously broken SU(2)xU(1) theory

\[ \mathcal{L}_{Higgs \ int} = \left( m_W^2 W_\mu^+ W^-_\mu + \frac{m_Z^2}{2} Z_\mu Z^\mu \right) \left( 1 + \frac{h}{v} \right)^2 - \frac{m_h^2}{2} - \frac{\xi}{3!} h^3 - \frac{\eta}{4!} h^4 \]

with
\[
\begin{align*}
m_W^2 & = \frac{1}{4} g^2 v^2 \\
m_Z^2 & = \frac{1}{4} (g^2 + g'^2) v^2 \\
m_h^2 & = 2 \lambda v^2 \\
\xi & = 3 \frac{m_h^2}{v} \\
\eta & = 6 \lambda = 3 \frac{m_h^2}{v^2}
\end{align*}
\]

where it is convenient to define the Weinberg mixing angle \( \theta_W \)

\[ \tan \theta_W = \frac{g'}{g} \]

and therefore
\[ \frac{m_W^2}{m_Z^2} = 1 - \sin^2 \theta_W \]
Spontaneously broken SU(2)xU(1) theory

From experiment we know

\[ m_W \simeq 80 \text{GeV} \]
\[ m_Z \simeq 91 \text{GeV} \]
\[ g \simeq 0.65 \]
\[ g' \simeq 0.35 \]

Hence we obtain \( v \simeq 246 \text{ GeV} \)

And for the couplings between V=W/Z bosons and the Higgs boson

\[ g_{hVV} = 2 \frac{m_V^2}{v^2} \]
\[ g_{hhhVV} = 2 \frac{m_V^2}{v^2} \]
\[ g_{hhh} = 3 \frac{m_h}{v^2} \]
\[ g_{hhhhh} = 3 \frac{m_h^2}{v^2} \]

We observe that the Higgs sector in the Standard Model is completely determined from the mass of the Higgs boson.
Theoretical constraints on the Higgs boson mass

Aim:
- Get a feeling how one can test if a theory is consistent
- How far can we stretch the EW theory until it does not make sense anymore?
- Example for the yet unobserved Higgs sector in the Standard Model, but techniques can be applied elsewhere

Content:
- Perturbativity & unitarity
- The triviality bound
- The vacuum stability bound
- The fine tuning constraints
Perturbative constraint & unitarity

The scattering of vector bosons at high energies is divergent due to their longitudinal polarization. Take $V = W$ or $Z$ traveling in the $z$-direction with 3-momentum magnitude $k$.

$$k^\mu = (E_k; \vec{k}) = (E_k; 0, 0, k)$$

with

$$E_k^2 = k^2 - m_V^2$$

The three polarization vectors are (resp. right handed, left handed and longitudinal):

$$\epsilon_+^\mu(\vec{k}) = \frac{1}{\sqrt{2}} (0; 1, i, 0)$$
$$\epsilon_-^\mu(\vec{k}) = \frac{1}{\sqrt{2}} (0; 1, -i, 0)$$
$$\epsilon_L^\mu(\vec{k}) = \frac{1}{m_V}(k; 0, 0, E_k)$$

which satisfy ($a, b = +, - , L$)

$$\vec{k}_\mu \epsilon^\mu_a(\vec{k}) = 0$$
$$\epsilon^\mu_a(\vec{k}) \epsilon^*_b(\vec{k}) = -\delta_{ab}$$
Perturbative constraint & unitarity

When $E_k >> m_V$ the longitudinal polarization is divergent. Diagrams with external vector bosons have divergent cross sections. Consider the process $W_L^+ W_L^- \rightarrow W_L^+ W_L^-$

(i) Four point interaction

(ii) Gauge exchange of photon/Z in the s- and t-channel

(iii) Higgs exchange in the s- and t-channel
Perturbative constraint & unitarity

The amplitude can be written as (S. Weinberg, Vol. 1, sec 3.7)

$$A = A^{(2)} s^2 + A^{(1)} s + A^{(0)}$$

From computations we learn that (when $s, t >> m_V^2, m_h^2$)

$$A^{(2)} \rightarrow 0$$
$$A^{(1)} \rightarrow 0$$
$$A^{(0)} \rightarrow -\frac{2m_h^2}{v^2} \approx -4\lambda$$

Perfect cancellation between the diagrams. But the amplitude remains proportional to the Higgs boson mass. If the Higgs boson mass is too large the theory becomes strongly interacting and we cannot perform expansions versus $\lambda$. 
Perturbative constraint & unitarity

At the loop level the process
\[ W^+ W^- \rightarrow (W W)_{\text{loop}} \rightarrow W^+ W^- \]
has an amplitude of
\[ \frac{2\lambda^2}{16\pi^2} \]

The one-loop amplitude becomes equal to the tree-level amplitude when \( \lambda \sim 16 \pi^2 \), hence the Electro-Weak theory should break down when \( m_h > 4.6 \text{ TeV} \).

More rigorous via partial wave analysis: \( m_h < 870 \text{ GeV} \)
When taking also the \( WW \rightarrow ZZ \) process into account: \( m_h < 710 \text{ GeV} \)
The triviality bound

The couplings should remain finite at all energy scales $Q$.

$$g_i = (0.41; 0.64; 1.2)$$

$$y_t = \sqrt{2} \frac{m_t}{v} \approx 1$$

$$\lambda = \frac{m_h^2}{2v^2}$$

Via the renormalization group equations we can evolve the couplings to higher scales $Q$.

$$\frac{dg_1}{dt} = \frac{41}{10} \frac{g_1^3}{16\pi^2}$$

$$\frac{dg_2}{dt} = -\frac{19}{6} \frac{g_2^3}{16\pi^2}$$

$$\frac{dg_3}{dt} = -7 \frac{g_3^3}{16\pi^2}$$

$$\frac{dy_t}{dt} = \frac{y_t}{16\pi^2} \left( -\frac{17}{20} g_1^2 - \frac{9}{4} g_2^2 - 8 g_3^2 + \frac{9}{2} y_t^2 \right)$$

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2} \left( 24\lambda^2 - \lambda \left( \frac{9}{5} g_1^2 + 9 g_2^2 + 12y_t^2 \right) + \frac{9}{8} \left( \frac{3}{25} g_1^4 + \frac{2}{5} g_1^2 g_2^2 + g_2^4 \right) - 6y_t^4 \right)$$

For large Higgs boson masses the term $\lambda^2$ dominates and after integration one obtains Landau pole or a limit on the value of $Q$ for which the theory is still valid.

$$Q_{LP} = m_h \exp \left( \frac{4\pi^2 v^2}{3m_h^2} \right)$$
The vacuum stability bound

When the Higgs boson mass is light the term $-6y_t^4$ will dominate:

$$\frac{d\lambda}{dt} \simeq -\frac{1}{16\pi^2} 6y_t^4$$

hence for higher scales $Q$ the value of $\lambda$ could become negative, hence the vacuum instable ($V<0$). With the constraint $\lambda(Q)>0$ for all values of $Q$ we obtain a lower limit on the Higgs boson mass. After integrating the part of the RGE which is $\lambda$ independent from $Q_0$ to $Q$ we obtain:

$$m_h^2 > \frac{v^2}{8\pi^2} \left( \frac{9}{8} \left( \frac{3}{25} g_1^4 + \frac{2}{5} g_1^2 g_2^2 + g_2^4 \right) - 6y_t^4 \right) \ln \left( \frac{Q}{Q_0} \right)$$

Hence a lower limit for the Higgs boson mass for a given $Q$ scale to keep the vacuum stable (without the presence of new physics phenomena beyond the Standard Model).

The full calculations at higher order (more loops) is done.
All together: theoretical bounds on the Higgs boson mass

If the Higgs boson is to be found at 60 GeV then this means the vacuum is instable in the absence of new physics. Only when the mass is between 130-180 GeV the vacuum can remain stable up to the Planck scale.
The fine-tuning constraint

The radiative corrections to the Higgs boson mass induce a fine tuning problem. At one loop, the integral can be cut-off at a momentum scale $\Lambda$

$$m_h^2 = (m_h^0)^2 + \frac{3\Lambda^2}{8\pi^2 v^2} \left( m_h^2 + 2m_W^2 + m_Z^2 - 4m_t^2 \right)$$

hence to cancel this we need $m_h^2 \sim (320 \text{ GeV})^2$

To cancel the radiative terms up to the GUT scale $\Lambda \sim 10^{16} \text{ GeV}$ we need to cancel up to 32 digits after the comma.
The fine-tuning constraint

Requesting up to 10% (or 1%) fine-tuning the allowed range for the validity of the Standard Model is reduced.
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Requesting up to 10% (or 1%) fine-tuning the allowed range for the validity of the Standard Model is reduced.

... why should there be only one Higgs doublet?
Two Higgs Doublet Models (2HDM)

In the Standard Model we have introduced only one complex Higgs doublet resulting into one physical Higgs boson field and masses for 3 vector bosons. There is however no experimental reason why we cannot have more than one Higgs doublet.

Let us introduce two complex Higgs doublet fields $\phi_1$ and $\phi_2$. The most general Higgs potential $V$ which spontaneously breaks $SU(2)_L \times U(1)_Y$ into $U(1)_{EM}$ is

$$V(\phi_1, \phi_2) = \lambda_1 (|\phi_1|^2 - v_1^2)^2 + \lambda_2 (|\phi_2|^2 - v_2^2)^2 + \lambda_3 \left[ (|\phi_1|^2 - v_1^2) + (|\phi_2|^2 - v_2^2) \right]^2 + \lambda_4 \left[ |\phi_1|^2 |\phi_2|^2 - (\phi_1^* T \phi_2)(\phi_2^* T \phi_1) \right] + \lambda_5 \left[ Re(\phi_1^* T \phi_2) - v_1 v_2 \cos \xi \right]^2 + \lambda_5 \left[ Im(\phi_1^* T \phi_2) - v_1 v_2 \sin \xi \right]^2$$

where the $\lambda_i$ values are real and the $\phi_i$’s are the Higgs fields with a minimum of the potential appearing at

$$\phi_1 = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}_1 = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ v_2 e^{i\xi} \end{pmatrix}$$
Two Higgs Doublet Models (2HDM)

When \( \sin \xi = 0 \) there is no CP violation in the Higgs sector, which we will force. The last two terms can be combined when \( \lambda_5 = \lambda_6 \) into

\[
\left| \phi_1^T \phi_2 - v_1 v_2 e^{i\xi} \right|^2
\]

where the \( e^{i\xi} \) term can be rotated away by redefining one of the \( \phi \) fields.

We develop the two doublets around the minimum of the potential. For this we parameterize the fields as

\[
\phi_1 = \begin{pmatrix} \phi_1^+ \\ v_1 + \eta_1 + i\chi_1 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} \phi_2^+ \\ v_2 + \eta_2 + i\chi_2 \end{pmatrix}
\]

where \( \eta_i \) is the CP-even part and \( \chi_i \) the CP-odd part. We put these fields in the potential and the mass terms appear.
Two Higgs Doublet Models (2HDM)

This results in the following relevant terms, grouped according to the CP-even, CP-odd and charged Higgs sectors:

\[
(\eta_1 \eta_2) \begin{pmatrix}
4(\lambda_1 + \lambda_3)v_1^2 + \lambda_5 v_2^2 & (4\lambda_3 + \lambda_5)v_1 v_2 \\
(4\lambda_3 + \lambda_5)v_1 v_2 & 4(\lambda_2 + \lambda_3)v_2^2 + \lambda_5 v_1^2
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}
\]

\[
\lambda_6 (\chi_1 \chi_2) \begin{pmatrix}
v_2^2 & -v_1 v_2 \\
-v_1 v_2 & v_1^2
\end{pmatrix}
\begin{pmatrix}
\chi_1 \\
\chi_2
\end{pmatrix}
\]

\[
\lambda_4 (\phi^-_1 \phi^-_2) \begin{pmatrix}
v_2^2 & -v_1 v_2 \\
-v_1 v_2 & v_1^2
\end{pmatrix}
\begin{pmatrix}
\phi^+_1 \\
\phi^+_2
\end{pmatrix}
\]

where these squared-mass terms can be obtained from

\[
M_{ij}^2 = \frac{1}{2} \frac{\partial^2 V(\phi_1, \phi_2)}{\partial \psi_i \partial \psi_j}
\]

with \( \psi_i \in \{ \phi_1^\pm, \phi_2^\pm, \eta_1, \eta_2, \chi_1, \chi_2 \} \)
Two Higgs Doublet Models (2HDM)

The physical eigenstates of the Higgs fields are obtained from a mixing between these fields \( \psi_i \in \{ \phi_1^\pm, \phi_2^\pm, \eta_1, \eta_2, \chi_1, \chi_2 \} \)

With a rotation of the eigenstates the squared-mass matrices can be diagonalized and the masses of the physical eigenstates can be determined.

For the CP-odd Higgs (mixing angle \( \beta \) with \( \tan \beta = v_2/v_1 \))

\[
M_A^2 = \lambda_6 (v_1^2 + v_2^2) \quad M_{G0}^2 = 0
\]

For the CP-even Higgs (mixing angle \( \alpha \))

\[
M_{H^0,h^0}^2 = \frac{1}{2}M_{11} + M_{22} \pm \sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2}
\]

For the charged Higgs (mixing angle \( \beta \))

\[
M_{H^\pm}^2 = \lambda_4 (v_1^2 + v_2^2) \quad M_{G^\pm}^2 = 0
\]

3 Goldstone bosons to give masses to W and Z bosons
Two Higgs Doublet Models (2HDM)

These relations depend on the values of $\lambda$ and the mixing angles, hence they can be inverted to write the $\lambda$ values as a functions of the masses and mixing angles.

These will fully define the potential. Hence this non-CP violating 2HDM Higgs sector has 6 free parameters:

$$M_{H^\pm}, M_{H^0}, M_{h^0}, m_{A^0}, \tan\beta, \alpha$$

The fermions can couple to these two Higgs doublet field in different ways:

- **Type-I 2HDM**: the field $\phi_2$ couples couples to both the up- and down-type fermions
- **Type-II 2HDM**: the field $\phi_1$ generates the masses for the down-type fermions, while the field $\phi_2$ generates the masses for the up-type quarks (this is the basis of the Higgs sector in the MSSM)

The couplings between the fermions and the neutral Higgs bosons are defined from the mixing angles $\alpha$ and $\beta$. 
Lecture summary

- Reminder of the mechanism of spontaneous symmetry breaking
- Applied on the EW symmetry of SU(2)xU(1)
- The yet to be observed Higgs sector of the Standard Model depends only on one parameter, the mass of the Higgs boson

- Diverse arguments can be invoked to put theoretically constraints on the Higgs boson mass ($m_H < 1 \text{ TeV}$)
- For the theory to be valid up to the Planck scale, the allowed range of the Higgs boson mass is very limited ($m_H \sim 130-180 \text{ GeV}$)
- When you do not “believe” that Nature has fine-tuned the parameters of the model, the allowed range is even vanishing or new physics has to appear at scale below $\Lambda \sim 10-100 \text{ TeV}$

- Maybe one Higgs doublet is not enough…
- 2-Higgs Doublet Models are the basis of supersymmetric models
- We have walked through the techniques needed to calculate the mass spectrum of the Higgs sector in a general 2HDM